The cocktail (highball) problem

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A simple analysis of the impact with the ground of two or more elastic balls in contact yields some interesting results.

A popular demonstration of the principles of conservation of linear momentum and energy involves dropping two contacting 'superballs' vertically to the ground. If the lower ball is more massive than the upper, kinetic energy is transferred to the latter with the result that its maximum height after the collision can be considerably greater than that of the original release point. In this article, a simple analysis of the dynamics of this process is presented, together with extensions to the case of three or more balls. The theoretical limits of the final velocity of the uppermost ball are calculated for several cases, with some unexpected results.

Two-ball problem

We let the masses of the lower and upper balls be m_1 and m_2 respectively with $m_1 > m_2$. They are dropped vertically from an original height h_0. It is assumed that h_0 is very much greater than the radii of the balls. The balls have a common downward velocity $v_0 = \sqrt{2gh_0}$ just before impact. If the lower ball makes an elastic collision with the ground, it will rebound with an upward speed of v_0. The subsequent collision between the two balls, also assumed to be perfectly elastic, is as shown in figure 1. If we take up as positive and apply conservation of linear momentum and kinetic energy, we obtain

$$m_1 v_0 - m_2 v_0 = m_2 v_2 + m_1 v_1. \qquad (1)$$

$$\tfrac{1}{2} m_1 v_0^2 + \tfrac{1}{2} m_2 v_0^2 = \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{2} m_1 v_1^2. \qquad (2)$$

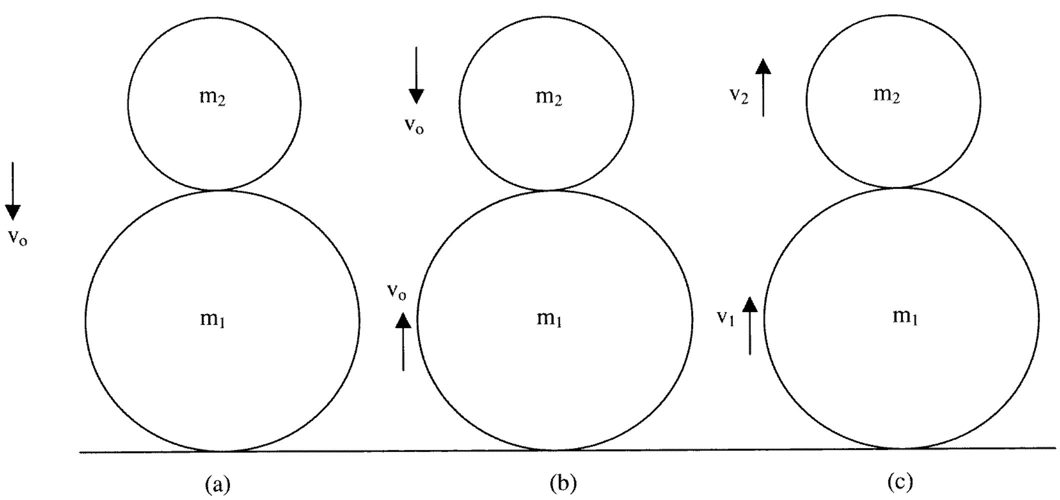

Figure 1. The two-ball problem.

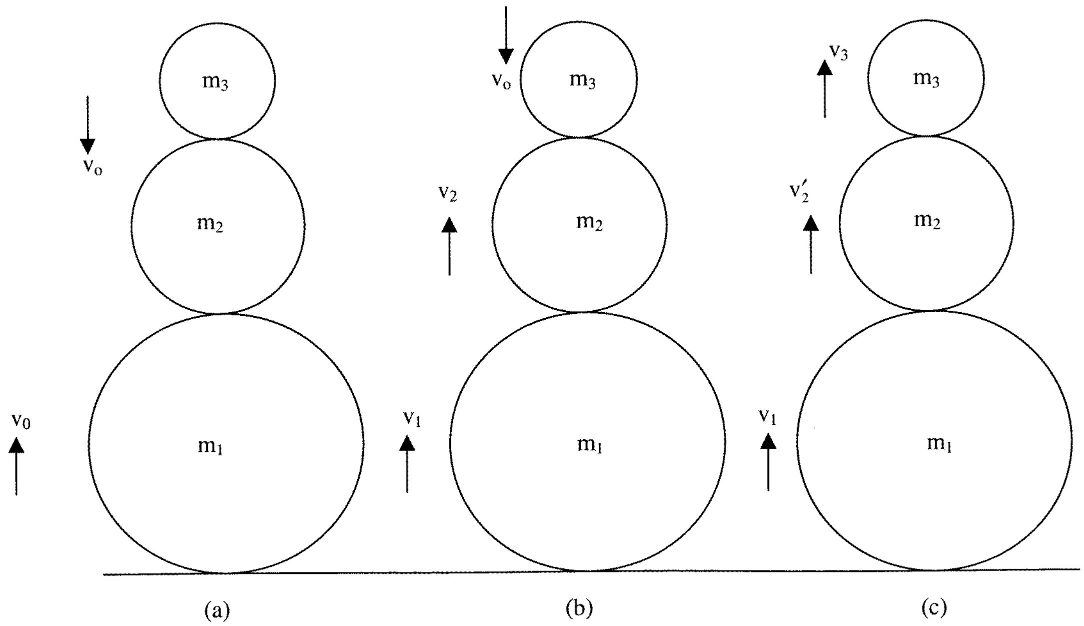

Figure 2. The three-ball problem.

From (1):

$$m_1(v_0 - v_1) = m_2(v_2 + v_0). \tag{3}$$

From (2):

$$\tfrac{1}{2}m_1(v_0^2 - v_1^2) = \tfrac{1}{2}m_2(v_2^2 - v_0^2)$$

or

$$m_1(v_0 - v_1)(v_0 + v_1) = m_2(v_2 - v_0)(v_2 + v_0). \tag{4}$$

Divide (4) by (3):

$$v_0 + v_1 = v_2 - v_0$$

or

$$2v_0 = v_2 - v_1. \tag{5}$$

Equation (5) is an example of the principle that for any one-dimensional elastic collision, the relative speed of approach equals the relative speed of separation. We will use the linear equations (1) and (5) to find the rebound speed of the upper ball. This results in considerable algebraic simplification compared with the direct use of equation (2), with its squared velocity terms.

From (5):

$$v_1 = v_2 - 2v_0. \tag{6}$$

Substitute (6) into (1):

$$m_1v_0 - m_2v_0 = m_2v_2 + m_1(v_2 - 2v_0).$$

Solve for v_2:

$$v_2 = (3m_1 - m_2)v_0/(m_1 + m_2). \tag{7}$$

We apply equation (7) to three special cases:

(a) if $m_1 = m_2$, $v_2 = v_0$ as expected
(b) if $m_1 = 3m_2$, $v_2 = 2v_0$
(c) if $m_1 \gg m_2$, $v_2 = 3v_0$.

Hence the limiting maximum height reached by the upper ball is $h_2 = 9h_0$.

Three-ball problem

The collisions are represented in figure 2. The one of main interest is between m_2 with initial upward speed v_2 given by equation (7) and m_3 with initial downward speed v_0. We use equations (1) and (5) with appropriate changes:

$$m_2v_2 - m_3v_0 = m_3v_3 + m_2v_2' \tag{1'}$$

$$v_2 + v_0 = v_3 - v_2'. \tag{5'}$$

From (5′):

$$v_2' = v_3 - v_2 - v_0. \tag{8}$$

Substitute (8) into (1′):

$$m_2 v_2 - m_3 v_0 = m_3 v_3 + m_2(v_3 - v_2 - v_0).$$

Use (7) to eliminate v_2 and solve for v_3:

$$v_3 = \frac{(7m_1m_2 - m_2^2 - m_1m_3 - m_2m_3)v_0}{(m_2+m_3)(m_1+m_2)}. \tag{9}$$

We apply equation (9) to three special cases:

(*a*) if $m_1 = m_2 = m_3$, $v_3 = v_0$ as expected
(*b*) if $m_1 = 3m_2 = 9m_3$, $v_3 = 3.5v_0$
(*c*) if $m_1 \gg m_2 \gg m_3$, $v_3 = 7v_0$.

Hence the limiting maximum height reached by the uppermost ball is $h_3 = 49h_0$.

Now suppose that the total mass of the three balls is constant (with $M = m_1 + m_2 + m_3$) and that a particular mass of the lightest ball is selected (with $\gamma = m_3/M$). What is the value of m_2/m_1 such that v_3 is a maximum, and how does v_3 depend on γ in this case?

We define $\beta = m_2/M$ and $\alpha = m_1/M$ with $\alpha + \beta + \gamma = 1$. Then from (9):

$$v_3 = \frac{(7\alpha\beta - \beta^2 - \alpha\gamma - \beta\gamma)v_0}{(\beta+\gamma)(\alpha+\beta)}. \tag{10}$$

We replace α by $(1 - \beta - \gamma)$ and treat β as a variable, with γ constant. After some manipulation, equation (10) may be written as follows:

$$v_3 = \frac{7\beta - 8\beta^2 - 7\beta\gamma - \gamma + \gamma^2}{(\beta+\gamma)(1-\gamma)}. \tag{11}$$

To find the condition for maximum v_3, we differentiate equation (11) and equate $\mathrm{d}v_3/\mathrm{d}\beta$ to zero. After considerable algebraic bookkeeping, during which many terms cancel, the following surprisingly simple quadratic equation emerges:

$$\beta^2 + 2\beta\gamma + \gamma(\gamma - 1) = 0. \tag{12}$$

Moreover, this equation factorizes:

$$(\beta + \gamma - \sqrt{\gamma})(\beta + \gamma + \sqrt{\gamma}) = 0.$$

The positive root gives

$$\beta = \sqrt{\gamma} - \gamma \tag{13a}$$

from which we find that

$$\alpha = 1 - (\beta + \gamma) = 1 - \sqrt{\gamma}. \tag{13b}$$

The required mass ratio is†

$$m_2/m_1 = \beta/\alpha = \sqrt{\gamma}. \tag{14}$$

Substitution of these optimal values of α and β (equations (13)) into equation (10) gives the maximum value of v_3 as follows:

$$v_3 = \frac{(7 - 9\sqrt{\gamma})v_0}{1 + \sqrt{\gamma}}. \tag{15}$$

We apply equation (15) to three special cases:

(*a*) if $\gamma = m_3/M = 0.04$,
then $m_2/m_1 = 0.2$ and $v_3 = 4.33v_0$
(*b*) if $\gamma = 0.01$,
then $m_2/m_1 = 0.1$ and $v_3 = 5.55v_0$
(*c*) if $\gamma = 0.0001$,
then $m_2/m_1 = 0.01$ and $v_3 = 6.84v_0$.

Clearly the last example is approaching the limiting value of $v_3 = 7v_0$ obtained earlier for the extreme case of $m_1 \gg m_2 \gg m_3$.

Multi-ball problem

We have shown that the maximum speed of the uppermost ball has the following limiting values:

$$v_1 = v_0; \quad v_2 = 3v_0; \quad v_3 = 7v_0.$$

Extending the analysis to four balls with $m_1 \gg m_2 \gg m_3 \gg m_4$ gives $v_4 = 15v_0$ with $h_4 = 225h_0$. It can be shown that in general

$$v_n = (2^n - 1)v_0 \tag{16}$$

where n is the number of balls.

If we were to drop the balls from the top of the CN Tower in Toronto, which has a height $h_0 = 560$ m, through frictionless air or a very long vacuum tube, we could ask how many balls would be required in order to give the uppermost one a rebound speed greater than that necessary to escape the gravitational pull of the Earth. The

† Often when a simple result such as equation (14) emerges from a complicated algebraic expression, the inference is that a different, perhaps more subtle, approach would give the answers more directly. Are there any suggestions from readers of *Physics Education*?

escape speed, which is independent of the mass of the ball, is given by the following well-known expression:

$$v_{esc} = \sqrt{2GM_E/R_E}$$

and has a value of 11 200 m s^{-1}. It is easily shown that

$$v_0 = \sqrt{2gh_0} = \sqrt{2 \times 9.8 \times 560} = 104.8\ \mathrm{m\,s^{-1}}$$

and for $n = 7$ we find $v_7 = (2^7 - 1)v_0 = 127v_0 = 13\,410\ \mathrm{m\,s^{-1}}$, provided that $m_n \ll m_{n-1}$, all collisions are elastic and all dissipative processes are neglected. Unfortunately, even if the mass ratio for adjacent balls were limited to 10^3, we find that to launch a microgram ball into space, the mass of m_1 would need to be about 10^9 kg, more than enough to destroy the foundations of the tower!

A cautionary note

A word of warning is in order for those instructors, especially males, planning to demonstrate these physical principles. In an item on the popular TV program *America's Funniest Home Videos*, a professor was filmed, presumably by a student in the audience, demonstrating the two-ball problem. He carefully positioned the balls so that their intended line of action was well away from the overhead lights, saying 'Now we won't do any damage'. Alas, on impact with the floor, the line between ball centres was not quite vertical. The smaller ball rebounded obliquely at great speed and hit the professor in a most sensitive spot, causing him to exclaim, in a falsetto voice, 'Maybe my last statement was somewhat optimistic!' It is recommended that the balls (please don't ask which ones!) be drilled and loosely assembled on a stiff wire, so that the direction of rebound motion is reasonably well controlled. In the worst possible scenario, it would be most unfortunate if the value of n in equation (16) above were inadvertently increased by one or two. . . .

Acknowledgment

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Further reading

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Another round of cocktails (highballs)

In a recent paper in this journal [1], a simple analysis of the impact with the ground of two or more contacting elastic balls was presented. It was shown that the maximum speed of the uppermost nth ball was given by

$$v_n = (2^n - 1)v_0$$

where $v_0 = (2gh_0)^{1/2}$, h_0 being the height of the original release point above ground level. This result is conditional on $m_n \ll m_{n-1}$, all collisions being perfectly elastic and all dissipative processes being neglected.

It has been pointed out [2] that the masses of the balls can also be arranged so that the total kinetic energy is delivered to the top ball, leaving all the others at rest at ground level (assuming their sizes are $\ll h_0$). We first consider the two-ball problem: the lower ball has mass m_1 and bounces from the floor with speed v_0. It collides with the second ball, mass m_2, moving downwards with speed v_0. If the lower ball is to be at rest after this collision, then the upper ball must have speed $2v_0$ in order to satisfy the condition that the relative speed of approach must equal the relative speed of separation for a one-dimensional elastic collision [1]. The relation between the masses is obtained by applying momentum conservation:

$$m_1v_0 - m_2v_0 = m_2 2v_0 + m_1 0$$

which gives

$$m_2 = m_1/3.$$

Similarly, the collision between m_2 and m_3 can be described as follows:

$$m_2 2v_0 - m_3v_0 = m_3 3v_0 + m_2 0.$$

Hence

$$m_3 = m_2/2 = m_1/6.$$

Extension to four balls gives the following result:

$$m_3 3v_0 - m_4v_0 = m_4 4v_0 + m_3 0$$

and so

$$m_4 = 3m_3/5 = m_1/10.$$

In general, if

$$m_n = m_1/[n(n+1)/2]$$

then the nth ball will have an upwards speed of nv_0 and will rise to a height of n^2h_0. Incidentally, the term in square brackets represents the number of balls in a close-packed equilateral triangle with n along each side. (Anyone for a game of snooker as we drink our cocktails?)

The kinetic energy of the nth ball is given by

$$E_{kn} = \frac{1}{2}m_nv_n^2 = \frac{1}{2}\frac{m_1}{[n(n+1)/2]}(nv_0)^2 = \frac{n}{n+1}m_1v_0^2.$$

For large n, it can be seen that E_{kn} approaches a limiting value of twice the initial kinetic energy of the bottom ball. The top ball increases speed proportionately with n, but its diminishing mass regulates the total energy it can obtain. It is also easy to relate the sum of all the masses, $M = \sum_1^n m_n$, to that of the bottom ball for large values of n: since the initial kinetic energy of all balls is transferred to the top one, we have

$$\frac{1}{2}Mv_0^2 = \frac{1}{2}m_nv_n^2 = \frac{n}{n+1}m_1v_0^2.$$

For $n \gg 1$, $M = 2m_1$. This result can also be obtained mathematically as follows [3]:

$$\begin{aligned} M &= m_1 + m_2 + \ldots + m_n \\ &= m_1\left(1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{2}{n(n+1)}\right). \end{aligned}$$

Each term can be expressed as the difference between two parts:

$$\frac{2}{n(n+1)} = \frac{2}{n} - \frac{2}{n+1}.$$

When the summation is made, all terms cancel except the first and last. Hence

$$M = m_1\left(2 - \frac{2}{n+1}\right) = 2m_1 \quad \text{as } n \to \infty.$$

In an earlier paper [1], it was shown that if the balls are dropped from the top of the CN Tower in Toronto (h_0 = 560 m), and if $m_n \ll m_{n-1}$, only seven balls are required to give the top one a speed greater than required to escape the gravitational field of the Earth, again assuming all collisions are elastic and all dissipative processes are neglected. However, even if the mass ratio for adjacent balls is limited to 1000, we found that to launch a microgram ball into space, the mass of the bottom ball would need to be 10^9 kg, so that this experiment would presumably be unacceptable to the owners of the tower and nearby residents. If, on the other hand, we use balls with the sequence of masses outlined above, although we would need 107 balls to achieve escape speed for the top one, the mass of the bottom ball would be only 5778 times that of the top one and the total mass of all balls just twice this value. In other words, to launch a 10 g ball into space, we would need a total mass of only 115 kg. Moreover, since all remaining balls are nominally at rest after the collisions, multiple launches could be readily made by replacing the top ball and taking the assembly back to the top of the tower using its fast elevator!

A more realistic demonstration involves five identical superballs, each of mass 10 g, say, which are weighted with different amounts of lead shot inserted symmetrically into small holes to give resultant masses of 10, 15, 25, 50 and 150 g. Alternatively, balls increasing in radii to give these mass ratios can be used. A commercial assembly under the name Ninja-Balls is available from scientific suppliers. The balls are also drilled vertically and loosely mounted on a stiff wire for safety reasons, as discussed earlier [1]. Dropping the assembly from a height of 2 m should result in the 10 g ball reaching a height close to the theoretical value of 50 m. Try it outside!

Helpful discussions with our colleagues Neil Isenor and Phil Eastman are gratefully acknowledged.

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References

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Interpolation: a lost art

Those of us of a certain age who grew up before the pocket calculator had several advantages over students today. Admittedly there were disadvantages as well. However, slide rules taught us the significance of significant figures (pun intended!). Using trigonometry and logarithmic tables gave us a facility in the use of interpolation. This also meant that we were comfortable with the ideas of estimation and approximation quite early in our studies. We learned to think in orders of magnitude and to judge the reasonableness of an answer. Students today, in contrast, believe in the absolute truth of calculators and cheerfully write down answers to seven places if the calculator shows seven places. The calculator has become a Delphic oracle—unfortunately the students cannot interpret its utterances.

Square roots offer a splendid opportunity to introduce the idea of interpolation to students. An important by-product is that it also allows us to show the usefulness of scientific notation. Many years ago I developed a technique to take square roots in my head. It seemed quite obvious to me, but several people have said that it is not and have asked me to share it. Moreover, as I examined the technique, I was able to prove a theorem that confirms its validity.

Let me first demonstrate the method. Consider a four-digit number, for example 6736. We first write it as 67.36×10^2. This reduces it to a more manageable form. The square root of the exponential term is 10. Since 67.36 lies between 64 and 81, its root must lie between 8 and 9. Now, 81 is 17 more than 64 and 67.36 is 3.36 more than 64. Let the root of 67.36 be $8 + \alpha$, where α is less than 1 and unknown. The basis of the method is to assume that

$$\alpha = 3.36/17 = 0.198. \tag{1}$$

(Note that the division may be done mentally.) In other words, the root lies between 8 and 9 in the same ratio as its square does between 64 and 81. We therefore assert the root of 67.36 to be 8.198. The correct value is 8.207, which is equivalent to an error of 9 parts in 8200 or 0.11%, and the root of 6736 is therefore 81.98.

Over the years I have found that the method works well, giving results to better than 1%. Let us now examine it and see if we can justify it. The square of a digit n is n^2. The square of $n + 1$ is

$$(n + 1)^2 = n^2 + 2n + 1. \tag{2}$$

An increase in n by 1 increases the square by $2n+1$. If $\alpha < 1$, what is the value of $(n + \alpha)^2$?

$$(n + \alpha)^2 = n^2 + 2\alpha n + \alpha^2. \tag{3}$$

Clearly $2\alpha n + \alpha^2$ must be a fraction β of $(2n + 1)$. This yields

$$2\alpha n + \alpha^2 = \beta(2n + 1) = 2\beta n + \beta. \tag{4}$$

Since α and β are both less than 1, we find to the first order that

$$\alpha \sim \beta. \tag{5}$$

This is the justification of equation (1).

As a second example take the five-digit number, 43 278. This may be written as 4.3278×10^4. Its root lies between 2 and 3×10^2. Applying equation (5), we find

$$\alpha = 0.3278/5 = 0.0658. \qquad (6)$$

Therefore the root of 43 278 is 2.0658×10^2. The correct value is 2.0803×10^2, an error of 0.7%.

Explaining the technique so that students can understand and use it does take time. The results are well worth it. With practice they are able to use the method themselves. They begin to understand interpolation and see the power of scientific notation. This leads immediately to better understanding of order of magnitude. They begin to know in advance the range in which the unknown root must lie. They also learn to appreciate the power of proof in mathematics. This is an appreciation many students lack. Discussing the meaning of equation (5) makes them aware that the method is not limited to specific numbers or ranges but applies quite generally.

My goal is to show that the calculator, though a useful tool, is no more than that. The mind is more powerful.

I wish to thank my colleagues, James Anton and Andrew Freda, of The Rivers School for their interest in the method and their encouragement to share it. They are both much aware of the mathematical limitations of students today and seek ways of replacing rote methods with understanding.

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What's in a bulb?

The article on the switching time of a 100 watt bulb (Menon V J and Agrawal D C 1999 *Phys. Educ.* **34** 34–6) is somewhat unrealistic in that it ignores heat losses from the filament of the bulb. It is easy to add terms for the thermal radiation from the surface of the filament, although it will then be necessary to integrate numerically. A graph of temperature against time will then be seen to reach a steady temperature. The time required is somewhat arbitrary because it depends on what temperature is taken to represent full brilliance, but it will be somewhat larger than the figure of 0.06 s stated in the article, probably quite near the value 0.10 s.

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