UNIVERSITETET I TRONDHEIM
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LØSNINGSFORSLAG:
KONTINUASJONSEKSAMEN I FAG 71527 RELATIVISTISK KVANTEMEKANIKK
Mandag 24. august 1987

Oppgave I.

1. Minner om at $A^0 = A_0, A_i = -A^i, \partial^0\phi = \partial_0\phi$:

   i) $C_1 = A_\mu A^\mu = A^0 A^0 - A^i A^i = A^0 A^0 - \vec{A}\cdot\vec{A}$

   ii) $C_2 = j_\mu A^\mu = j^0 A^0 - j^i A^i = j^0 A^0 - \vec{j}\cdot\vec{A}$

   iii) $C_3 = (\partial_\mu A_\nu)(\partial^\mu A^\nu) = (\partial_0 A_\nu)(\partial_0 A^\nu) - (\partial_i A_\nu)(\partial_i A^\nu)$
   $= (\partial_0 A^0)(\partial_0 A^0) - (\partial_0 A^j)(\partial_0 A^j) - (\partial_i A^0)(\partial_i A^0) + (\partial_i A^j)(\partial_i A^j)$

   iv) $C_4 = (\partial_\mu A_\nu)(\partial^\nu A^\mu) = (\partial_0 A_\nu)(\partial^\nu A_0) + (\partial_i A_\nu)(\partial^\nu A_i)$
   $= (\partial_0 A^0)(\partial_0 A^0) + (\partial_0 A^j)(\partial_j A^0) + (\partial_i A^0)(\partial_0 A^i) + (\partial_i A^j)(\partial_j A^i)$

   De to midterste leddene i siste uttrykk er like.

2. Vi setter inn uttrykkene fra pkt 1 og får

$$\mathcal{L} = -\frac{1}{2}C_3 + \frac{1}{2}C_4 + \frac{1}{2}m^2 C_1 - C_2$$

$$= \frac{1}{2}(\partial_0 A^i)(\partial_0 A^i) + (\partial_0 A^i)(\partial_i A^0) + \frac{1}{2}(\partial_i A^0)(\partial_i A^0) - \frac{1}{2}(\partial_i A^j)(\partial_i A^j - \partial_j A^i)$$

$$+ \frac{1}{2}m^2(A^0A^0 - A^iA^i) - j^0A^0 + j^iA^i \quad .$$

3.

$\Pi_0 = 0$ siden $\mathcal{L}$ ikke avhenger av $(\partial_0 A^0)$.

$\Pi_i = \partial_0 A^i + \partial_i A^0$.

4. Ser på tilfellene $\nu = 0$ og $\nu = i$ separat:

$\nu = 0$: Vi finner $\partial\mathcal{L}/\partial(\partial_0 A^0) = 0$, $\partial\mathcal{L}/\partial(\partial_j A^0) = \partial_j A^0 + \partial_0 A^j$ og $\partial\mathcal{L}/\partial A^0 = m^2 A^0 - j^0$.
Altså

$$\partial_j [\partial_j A^0 + \partial_0 A^j] = m^2 A^0 - j^0$$

eller

$$(-\Delta + m^2) A^0 = j^0 + \partial_0 (\partial_j A^j) \quad .$$

$\nu = i$: Vi finner $\partial\mathcal{L}/\partial(\partial_0 A^i) = \partial_0 A^i + \partial_i A^0$, $\partial\mathcal{L}/\partial(\partial_0 A^i) = -(\partial_j A^i - \partial_i A^j)$ og $\partial\mathcal{L}/\partial A^i = -m^2 A^i + j^i$.
Altså

$$\partial_0 \partial_0 A^i - \partial_j \partial_j A^i + \partial_0 \partial_i A^0 + \partial_i \partial_j A^j = -m^2 A^i + j^i$$

eller

$$(\partial_\mu \partial^\mu + m^2) A^i = j^i - \partial_i (\partial_0 A^0 + \partial_j A^j) \quad .$$

Ligningene for $\nu = 0$ og $\nu = i$ kan sammenfattes på kovariant form som

$$(\partial_\mu \partial^\mu + m^2) A^\nu = j^\nu + \partial^\nu (\partial_\mu A^\mu)$$

5.

i) I den følgende regningen må vi ha i minne at $\delta^{ij} = -\eta^{ij}$, dvs at $\delta^{ij} g_j = -\eta^{ij} g_j = -g^i$ osv. Vi setter så inn ansatzen for G^{ij}

$$\begin{aligned}
& G^{ij} [(-\Delta + m^2)\delta_{jk} + \partial_j \partial_k] \\
&= [A\delta^{ij} + B\partial^i \partial^j][(-\Delta + m^2)\delta_{jk} + \partial_j \partial_k] \\
&= A(-\Delta + m^2)\delta^i_k - A\partial^i \partial_k - B(-\Delta + m^2)\partial^i \partial_k - B\Delta\partial^i \partial_k \\
&= A(-\Delta + m^2)\delta^i_k - [A + m^2 B]\partial^i \partial_k = \delta^i_k
\end{aligned}$$

For at siste likhet skal stemme må $A(-\Delta + m^2) = 1$ og $A + Bm^2 = 0$. Disse ligningene har løsningene $A = (-\Delta + m^2)^{-1}$ og $B = -m^{-2}(-\Delta + m^2)^{-1}$.
Følgelig:

$$G^{ij} = \frac{1}{-\Delta + m^2} [\delta^{ij} - m^{-2} \partial^i \partial^j]$$

ii) Vi bruker resultatet fra pkt i) og får

$$\begin{aligned}
f^i &= G^{ij} [(-\Delta + m^2) g_j + (\partial_j h)] \\
&= \frac{1}{-\Delta + m^2} [\delta^{ij} - m^{-2} \partial^i \partial^j][(-\Delta + m^2) g_j + (\partial_j h)] \\
&= [-g^i - m^{-2} \partial^i (\partial^j g_j)] + \frac{1}{-\Delta + m^2} [-(\partial^i h) + m^{-2} \Delta (\partial^i h)] \\
&= [g_i - m^{-2} \partial_i (\partial_j g_j) + m^{-2} (\partial_i h)]
\end{aligned}$$

6. Vi har funnet at $\Pi_i = (\partial_0 A^i) + (\partial_i A^0)$. Derav

$$(-\Delta+m^2)\Pi_i = (-\Delta+m^2)(\partial_0 A^i) + \partial_i[(-\Delta+m^2)A^0]$$
$$= (-\Delta+m^2)\delta_{ij}(\partial_0 A^j) + (\partial_i j^0) + \partial_i\partial_j(\partial_0 A^j) \ ,$$

der vi i siste likhet har satt inn uttrykket for $(-\Delta+m^2)A^0$ fra pkt 4. Overstående ligning kan skrives om til

$$[(-\Delta+m^2)\delta_{ij} + \partial_i\partial_j](\partial_0 A^j) = (-\Delta+m^2)\Pi_i - (\partial_i j^0)$$

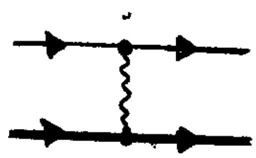
som er presis ligningen i pkt 5.ii), med $f^j = (\partial_0 A^j)$, $g_i = \Pi_i$ og $h = -j^0$. Løsningen blir derfor

$$(\partial_0 A^i) = \Pi_i - m^{-2}\partial_i\partial_j\Pi_j - m^{-2}(\partial_i j^0) \ .$$

Oppgave II.

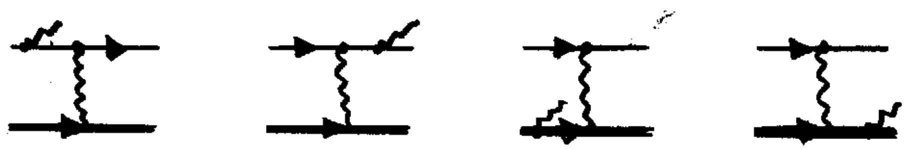
1.

i) $e^-\mu^- \to e^-\mu^-$

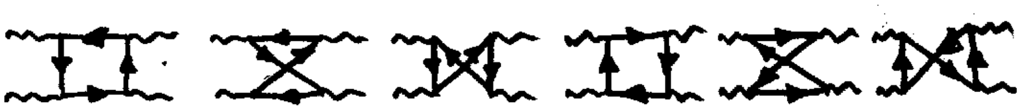


ii) $e^-\gamma \to e^-\gamma$

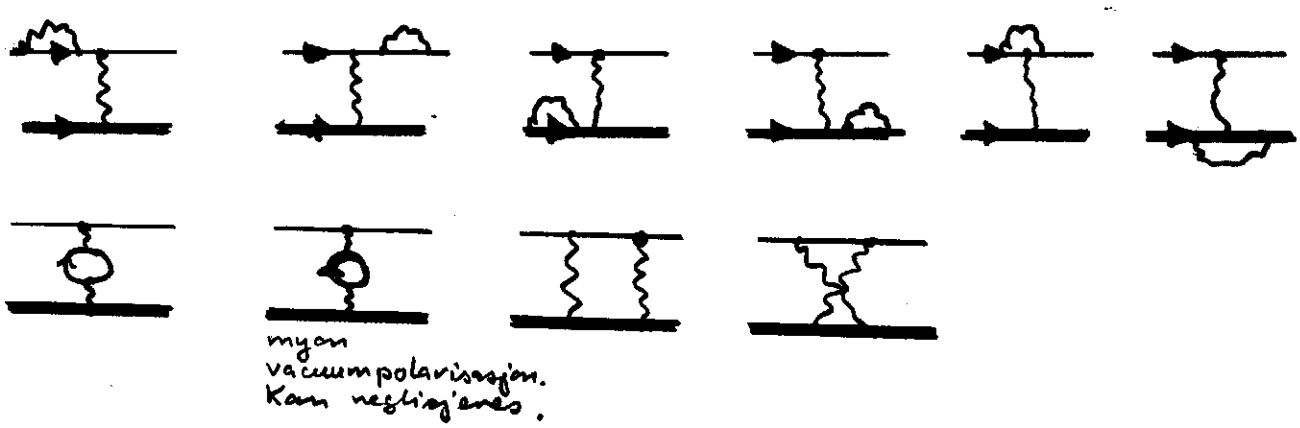
iii) $e^+\mu^- \to e^+\mu^-\gamma$



iv) $\gamma\gamma \to \gamma\gamma$



2. Vi har 9 diagrammer (foruten renormaliserings-bidrag og f.eks myon vacuumpolarisasjon):



3. Vi har to diagrammer og kan skrive $T_{fi} = T_{fi}^{(1)} + T_{fi}^{(2)}$, med

$$i\sqrt{4p_2^0 p_2'^0}\, T_{fi}^{(1)} = \frac{-ie^2}{q^2 - m^2 + i\epsilon} [\bar{u}(p_1', s_1') \not\epsilon^* (p_2', r_2') (\not q + m) \not\epsilon (p_2, r_2) u(p_1, s)]$$

$$i\sqrt{4p_2^0 p_2'^0}\, T_{fi}^{(2)} = \frac{-ie^2}{q'^2 - m^2 + i\epsilon} [\bar{u}(p_1', s_1') \not\epsilon (p_2, r_2) (\not q' + m) \not\epsilon^* (p_2', r_2') u(p_1, s)]$$

der $q = p_1 + p_2$ og $q' = p_1 - p_2'$.

4.

i) Vi ekspanderer først etter massen m:

$$S_1 = Tr[(\not p_1 + m)\gamma_\mu (\not q + m)\gamma_\nu (\not p_1' + m)\gamma^\nu (\not q + m)\gamma^\mu]$$
$$= S_{10} + m^2 (S_{11} + S_{12} + S_{13} + S_{14} + S_{15} + S_{16}) + m^4 S_{17}$$

der

$$S_{10} = Tr[\not p_1 \gamma_\mu \not q \gamma_\nu \not p_1' \gamma^\nu \not q \gamma^\mu] = 4\mathrm{Tr}\,(\not q \not p_1' \not q \not p_1)$$
$$= 32(p_1 q)(p_1' q) - 16(p_1 p_1')\, q^2 .$$

Her har vi først benyttet at sporet er syklisk, deretter identitetene $\gamma_\nu \not p_1' \gamma^\nu = -2\not p_1'$, $\gamma^\mu \not p_1 \gamma_\mu = -2\not p_1$ og tilsist uttrykket for spor over fire γ-matriser. Ved tilsvarende regning finner vi

$$S_{11} = \mathrm{Tr}\,[\gamma_\mu \gamma_\nu \not p_1' \gamma^\nu \not q \gamma^\mu] = -32(p_1' q)$$
$$S_{12} = \mathrm{Tr}\,[\gamma_\mu \not q \gamma_\nu \gamma^\nu \not q \gamma^\mu] = 64 q^2$$
$$S_{13} = \mathrm{Tr}\,[\gamma_\mu \not q \gamma_\nu \not p_1' \gamma^\nu \gamma^\mu] = -32(p_1' q)$$
$$S_{14} = \mathrm{Tr}\,[\not p_1 \gamma_\mu \gamma_\nu \gamma^\nu \not q \gamma^\mu] = -32(p_1 q)$$
$$S_{15} = \mathrm{Tr}\,[\not p_1 \gamma_\mu \gamma_\nu \not p_1' \gamma^\nu \gamma^\mu] = 16(p_1 p_1')$$
$$S_{16} = \mathrm{Tr}\,[\not p_1 \gamma_\mu \not q \gamma_\nu \gamma^\nu \gamma^\mu] = -32(p_1 q)$$
$$S_{17} = \mathrm{Tr}\,[\gamma_\mu \gamma_\nu \gamma^\nu \gamma^\mu] = 64$$

Altså

$$S_1 = 32(p_1 q)(p_1' q) - 16(p_1 p_1') q^2 + 16m^2 \{(p_1 p_1') - 4(p_1 q) - 4(p_1' q) + 4q^2\} + 64m^4 .$$

ii) Vi ekspanderer først etter massen m:

$$S_2 = \mathrm{Tr}\,[(\not p_1 + m)\gamma_\mu (\not q + m)\gamma_\nu (\not p_1' + m)\gamma^\mu (\not q' + m)\gamma^\nu]$$
$$= S_{20} + m^2 \{S_{21} + S_{22} + S_{23} + S_{24} + S_{25} + S_{26}\} + m^4 S_{27} ,$$

der

$$S_{20} = \mathrm{Tr}\,[\,(\not p_1 + m)\gamma_\mu\,(\not q + m)\gamma_\nu\,(\not p_1' + m)\gamma^\mu\,(\not q' + m)\gamma^\nu\,]$$
$$= -32(p_1p_1')(qq')$$
$$S_{21} = \mathrm{Tr}\,[\,(\gamma_\mu\gamma_\nu\not p_1'\gamma^\mu\not q'\gamma^\nu\,] = 16(p_1'q')$$
$$S_{22} = \mathrm{Tr}\,[\,(\gamma_\mu\not q\gamma_\nu\gamma^\mu\not q'\gamma^\nu\,] = 16(qq')$$
$$S_{23} = \mathrm{Tr}\,[\,(\gamma_\mu\not q\gamma_\nu\not p_1'\gamma^\mu\gamma^\nu\,] = 16(p_1'q)$$
$$S_{24} = \mathrm{Tr}\,[\not p_1\gamma_\mu\gamma_\nu\gamma^\mu\not q'\gamma^\nu\,] = 16(p_1q)$$
$$S_{25} = \mathrm{Tr}\,[\not p_1\gamma_\mu\gamma_\nu\not p_1'\gamma^\mu\gamma^\nu\,] = 16(p_1p_1')$$
$$S_{26} = \mathrm{Tr}\,[\not p_1\gamma_\mu\not q\gamma_\nu\gamma^\mu\gamma^\nu\,] = 16(p_1q)$$
$$S_{27} = \mathrm{Tr}\,[\gamma_\mu\gamma_\nu\gamma^\mu\gamma^\nu\,] = -32$$

Altså

$$S_2 = -32(p_1p_1')(qq') + 16m^2\{(p_1p_1') + (p_1q) + (p_1q')$$
$$+ (p_1'q) + (p_1'q') + (qq')\} - 32m^4 \quad .$$

5.

Vi regner først ut de skalarproduktene som inngår:

$$(p_1q) = (p_1'q) = \frac{1}{2}(s+m^2) \quad , \quad (p_1q') = (p_1'q') = \frac{1}{2}(u+m^2) \quad ,$$
$$(p_1p_1') = \frac{1}{2}(s+u) \quad , \quad q^2 = s \quad , \quad q'^2 = u \quad , \quad (qq') = m^2 \quad .$$

Innsatt i uttrykkene over gir dette

$$S_1 = -8su + (24s + 8u)m^2 + 8m^4$$
$$S_2 = 8(s+u)m^2 + 16m^4$$