UNIVERSITETET I TRONDHEIM
NORGES TEKNISKE HØGSKOLE
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LØSNINGSFORSLAG:
EKSAMEN I FAG 71527 RELATIVISTISK KVANTEMEKANIKK
Mandag 25. mai 1987

Oppgave I.

1.i) Mandelstam-invariantene

$$s = (p_1+p_2)^2 = (p_1'+p_2')^2 = m_1^2+m_2^2+2p_1\cdot p_2 = m_1'^2+m_2'^2+2p_1'\cdot p_2'$$

$$t = (p_1-p_1')^2 = (p_2'-p_2)^2 = m_1^2+m_1'^2-2p_1\cdot p_1' = m_2'^2+m_2^2-2p_2'\cdot p_2$$

$$u = (p_1-p_2')^2 = (p_2-p_1')^2 = m_1^2+m_2'^2-2p_1\cdot p_2' = m_2^2+m_1'^2-2p_2\cdot p_1'$$

ii) Vi adderer fra ligningene over:

$$s+t+u = (p_1+p_2)^2+(p_1-p_1')^2+(p_1-p_2')^2 =$$
$$m_1^2+m_2^2+m_1^2+m_1'^2+m_1^2+m_2'^2+2p_1\cdot(p_2-p_1'-p_2') \ .$$

Nå er $p_2-p_1'-p_2' = -p_1$, så siste ledd er lik $-2p_1\cdot p_1 = -2m_1^2$. Altså:

$$s+t+u = m_1^2+m_2^2+m_1'^2+m_2'^2 \ .$$

2.i) Siden $\vec{p}_1+\vec{p}_2 = 0$ blir $s = (p_1^0+p_2^0)^2 = m_1^2+m_2^2+2p^2+2\sqrt{(m_1^2+p^2)(m_2^2+p^2)}$. Flytter over og kvadrerer:

$$4(m_1^2+p^2)(m_2^2+p^2) = [2p^2-(s-m_1^2-m_2^2)]^2 \Rightarrow$$

$$p = \left(\frac{s^2-2(m_1^2+m_2^2)s+(m_1^2-m_2^2)^2}{4s}\right)^{1/2}$$

Tilsvarende ligning for p' fås ved å bytte til merkede størrelser:

$$p' = \left(\frac{s^2-2(m_1'^2+m_2'^2)s+(m_1'^2-m_2'^2)^2}{4s}\right)^{1/2} .$$

ii) Vi har

$$t = m_1^2 + m_1'^2 - 2p_1^0 p_2'^0 + 2\vec{p}_1\cdot\vec{p}_2 = m_1^2 + m_1'^2 - 2\sqrt{(m_1^2+p^2)(m_2^2+p'^2)} + 2pp'\cos\theta\,.$$

Nå er $p_1^0 = (s+m_1^2-m_2^2)/2\sqrt{s}$ og $p_1'^0 = (s+m_1'^2-m_2'^2)/2\sqrt{s}$. De tre første leddene kan derfor uttrykkes ved s som

$$m_1^2 + m_2'^2 - 2p_1^0 p_2'^0 = -\frac{s^2-(m_1^2+m_2^2+m_1'^2+m_2'^2)s+(m_1^2-m_2^2)(m_1'^2-m_2'^2)}{2s}\,,$$

og $p(s)$, $p'(s)$ er gitt over.

3.i) Siden $p_1+p_2 = (p_1^0+m_2, \vec{p}_1)$, der $p_1^0 = E$, blir

$$s = (E+m_2)^2 - \vec{p}_1^{\,2} = E^2 + 2Em_2 + m_2^2 - (E^2-m_1^2) = 2Em_2 + m_1^2 + m_2^2\,.$$

ii) Vi har

$$t = (p_1^0 - p_1'^0)^2 - (\vec{p}_1 - \vec{p}_1')^2\,.$$

Nå er $p_1^0 - p_1'^0 = \Delta E$ ved definisjon, og $(\vec{p}_1-\vec{p}_1')^2 = \vec{p}_2'^2 = p_2'^{02} - m_2'^2 = (m_2+\Delta E)^2 - m_2'^2$ ved impuls- og energi-konservering. Altså:

$$t = -(2m_2\Delta E + m_2^2 - m_2'^2)\,.$$

iii) Største energioverføring:

$$\Delta E_{\max} = \frac{m_2'^2 - m_2^2 - t_{\min}}{2m_2}\,,$$

der $t_{\min}$ finnes fra pkt.2ii for $\cos\theta = -1$:

$$-t_{\min} = \frac{s^2-(m_1^2+m_2^2+m_1'^2+m_2'^2)s+(m_1^2-m_2^2)(m_1'^2-m_2'^2)}{2s} + 2p(s)p'(s)\,.$$

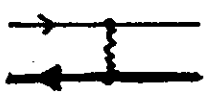
4. De forskjellige tverrsnittene er relatert ved

$$\frac{d\sigma}{dt}(s,t) = \frac{1}{2pp'}\frac{d\sigma}{d\cos\theta}(s,t) = \frac{1}{2m_2}\frac{d\sigma}{d\Delta E}(s,t)\,.$$

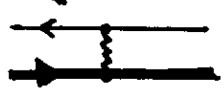
5. Feynmandiagrammene for laveste ordens prosesser:

i) $e^-\mu^- \to e^-\mu^-$

ii) $e^-\mu^+ \to e^-\mu^+$



iii) $e^+\mu^- \to e^+\mu^-$

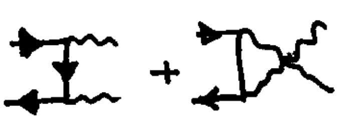


iv) $e^+\mu^+ \to e^+\mu^+$

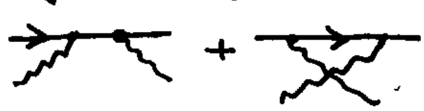
v) $e^-e^+ \to \mu^-\mu^+$

vi) $\mu^-\mu^+ \to e^-e^+$

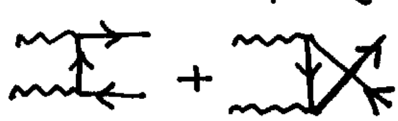
vii) $e^-e^+ \to \gamma\gamma$



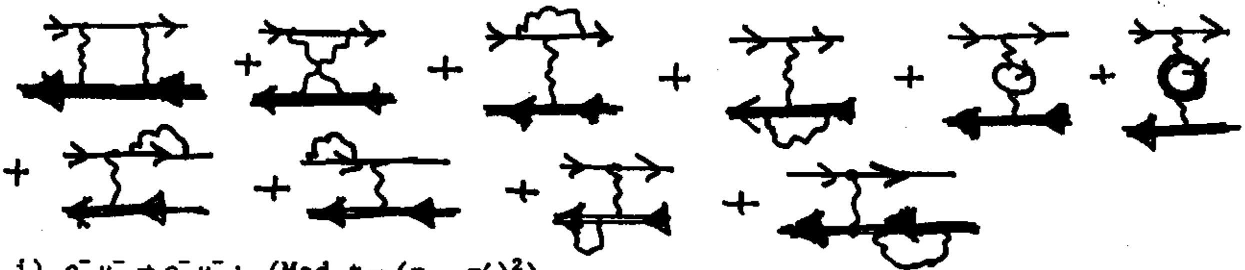
iix) $e^-\gamma \to e^-\gamma$



ix) $\gamma\gamma \to e^-e^+$



6. Feynmandiagrammene for neste ordens korreksjoner til prosessen $e^- \mu^+ \to e^- \mu^+$:



7.i) $e^- \mu^- \to e^- \mu^-$: (Med $t = (p_1 - p_1')^2$)

$$iT_{fi} = \frac{-i\eta_{\mu\nu}}{t} [\bar{u}(p_1', s_1')(-ie\gamma^\mu)u(p_1, s_1)]\, [\bar{u}(p_2', s_2')(-ie\gamma^\nu)u(p_2, s_2)]$$

ii) $e^- \mu^+ \to e^- \mu^+$:

$$iT_{fi} = \frac{-i\eta_{\mu\nu}}{t} [\bar{u}(p_1', s_1')(-ie\gamma^\mu)u(p_1, s_1)]\, [\bar{v}(p_2, s_2)(-ie\gamma^\nu)v(p_2', s_2')]$$

8. Vi kan skrive

$$\frac{1}{4} \sum_{s_1 s_2} \sum_{s_1' s_2'} \left| [\bar{u}(p_1', s_1')\gamma^\mu u(p_1, s_1)]\, [\bar{u}(p_2', s_2')\gamma^\nu u(p_2, s_2)]\eta_{\mu\nu} \right|^2$$

$$= \frac{1}{64 p_1^0 p_2^0 p_1'^0 p_2'^0} P^{(e^-)\mu\nu} P^{(\mu^-)}_{\mu\nu}$$

og

$$\frac{1}{4} \sum_{s_1 s_2} \sum_{s_1' s_2'} \left| [\bar{u}(p_1', s_1')\gamma^\mu u(p_1, s_1)]\, [\bar{v}(p_2, s_2')\gamma^\nu u(p_2', s_2)]\eta_{\mu\nu} \right|^2$$

$$= \frac{1}{64 p_1^0 p_2^0 p_1'^0 p_2'^0} P^{(e^-)\mu\nu} P^{(\mu^+)}_{\mu\nu}$$

med

$$P^{(X^\mp)}_{\mu\nu} = \mathrm{Tr}\,[(\not{p}_2 \pm m_X)\gamma_\mu (\not{p}_2' \pm m_X)\gamma_\nu] = \mathrm{Tr}\, \not{p}_2 \gamma_\mu \not{p}_2' \gamma_\nu + m_X^2 \mathrm{Tr}\, \gamma_\mu \gamma_\nu$$

siden sporet av tre γ-matriser er null. Altså blir

$$P^{(\mu^-)}_{\mu\nu} = P^{(\mu^+)}_{\mu\nu} \quad \text{og} \quad P^{(e^-)\mu\nu} = P^{(e^+)\mu\nu} .$$

Siden de øvrige (kinematiske) faktorene er de samme følger det av dette at

$$\frac{d\bar{\sigma}^{e^-\mu^- \to e^-\mu^-}}{d\cos\theta} = \frac{d\bar{\sigma}^{e^-\mu^+ \to e^-\mu^+}}{d\cos\theta} = \frac{d\bar{\sigma}^{e^+\mu^- \to e^+\mu^-}}{d\cos\theta} = \frac{d\bar{\sigma}^{e^+\mu^+ \to e^+\mu^+}}{d\cos\theta} .$$

9.i) Vi regner ut

$$P^{(e)\mu\nu} = \mathrm{Tr}\,\not p_1\gamma^\mu\not p_1'\gamma^\nu + m_e^2\,\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4[p_1^\mu p_1'^\nu + p_1^\nu p_1'^\mu + \eta^{\mu\nu}(m_e^2 - p_1\cdot p_1')]$$
$$P^{(\mu)}_{\mu\nu} = \mathrm{Tr}\,\not p_2\gamma_\mu\not p_2'\gamma_\nu + m_\mu^2\,\mathrm{Tr}\,\gamma_\mu\gamma_\nu = 4[p_{2\mu}p_{2\nu}' + p_{2\nu}p_{2\mu}' + \eta_{\mu\nu}(m_\mu^2 - p_2\cdot p_2')]\ .$$

Utfører kontraksjonen, og uttrykker skalarproduktene ved s og t ved hjelp av resultatene fra pkt.1:

$$P^{(e)\mu\nu}P^{(\mu)}_{\mu\nu} = 32[(p_1\cdot p_2)(p_1'\cdot p_2') + (p_1\cdot p_2')(p_1'\cdot p_2) - m_e^2(p_2\cdot p_2') - m_\mu^2(p_1\cdot p_1') + 2m_e^2m_\mu^2]$$
$$= 8[(s-m_e^2-m_\mu^2)^2 + (s+t-m_e^2-m_\mu^2)^2 - 2m_e^2(2m_\mu^2-t) - 2m_\mu^2(2m_e^2-t) + 8m_e^2m_\mu^2]$$
$$= 16[(s-m_e^2-m_\mu^2)^2 + st + \frac{1}{2}t^2]$$

Vi setter så $(p_1^0p_2^0p_1'^0p_2'^0)\,|T_{fi}|^2 = e^4P^{(e)\mu\nu}P^{(\mu)}_{\mu\nu}/64t^2$ inn i uttrykket for spredningstverrsnittet, integrerer over asimuth-vinkelen ϕ, og setter $d\cos\theta = dt/2p(s)p'(s)$. Dette gir

$$d\bar\sigma = \frac{1}{(p_1^0+p_2^0)^2}\,\frac{p'}{p}\,\frac{e^4}{64t^2}\,P^{(e)\mu\nu}P^{(\mu)}_{\mu\nu}\,\frac{dt}{4\pi pp'}$$
$$= \frac{\pi\alpha^2}{16\,p^2s\,t^2}\,P^{(e)\mu\nu}P^{(\mu)}_{\mu\nu}\,dt$$

ved bruk av $e^4 = 16\pi^2\alpha^2$ og $(p_1^0+p_2^0)^2 = s$. Videre er $p^2 = [s^2-2(m_\mu^2+m_e^2)s+(m_\mu^2-m_e^2)^2]/4s$, så vi får tilslutt

$$\frac{d\bar\sigma^{e^+\mu^-\to e^+\mu^-}}{dt} = \frac{4\pi\alpha^2}{t^2}\,\frac{(s-m_\mu^2-m_e^2)^2+st+\frac{1}{2}t^2}{s^2-2(m_\mu^2+m_e^2)s+(m_\mu^2-m_e^2)^2}\ .$$

ii) Vi bruker resultatene fra pkt.3 og 4 for å transformere dette uttrykket til variable i myonets labsystem:

$$\frac{d\bar\sigma^{e^+\mu^+\to e^+\mu^+}}{d\Delta E} = 2m_\mu\,\frac{d\bar\sigma^{e^+\mu^-\to e^+\mu^-}}{dt}(s = 2m_\mu E + m_e^2 + m_\mu^2\,,\ t = -2m_\mu\Delta E) =$$
$$\pi\alpha^2\,\frac{2E^2m_\mu^2 - (2Em_\mu^2 + m_\mu^3 + m_\mu^2m_e)\Delta E + 2m_\mu^2\Delta E^2}{m_\mu^3\,\Delta E^2(E^2-m_e^2)}\ .$$

Oppgave II.

1.i) Lagrangetettheten:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A^\nu)(\partial^\mu A_\nu) - j_\mu A^\mu =$$

$$-\frac{1}{2}\frac{\partial A^0}{\partial t}\frac{\partial A^0}{\partial t} + \frac{1}{2}\frac{\partial \vec{A}}{\partial t}\cdot\frac{\partial \vec{A}}{\partial t} + \frac{1}{2}\vec{\nabla}A^0\cdot\vec{\nabla}A^0 - \frac{1}{2}\sum_i \vec{\nabla}A^i\cdot\vec{\nabla}A^i - j^0 A^0 + \vec{j}\cdot\vec{A}\ .$$

De kanonisk konjugerte impulstetthetene:

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{A}^0} = -\dot{A}^0 \quad , \quad \Pi_i = \frac{\partial \mathcal{L}}{\partial \dot{A}^i} = +\dot{A}^i \quad ,$$

dvs. $\Pi_\mu = -\dot{A}_\mu$.

iii) Hamiltontettheten:

$$\mathrm{H} = \Pi_\mu \dot{A}^\mu - \mathcal{L} = -\Pi_\mu \Pi^\mu + \frac{1}{2}\partial_0 A^\nu \partial_0 A_\nu - \frac{1}{2}\vec{\nabla}A^\nu\cdot\vec{\nabla}A_\nu + j_\mu A^\mu =$$

$$-\frac{1}{2}\Pi_\mu \Pi^\mu - \frac{1}{2}\vec{\nabla}A^\nu\cdot\vec{\nabla}A_\nu + j_\mu A^\mu \quad .$$

2.i) De kanoniske kommutatorene

$$[\Pi_\mu(\vec{r}), A_\nu(\vec{r}')] = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{d^3k'}{(2\pi)^{3/2}}\beta(k)\alpha(k')\times$$

$$[a(k)_\mu e^{i\vec{k}\cdot\vec{r}} - a(k)^\dagger_\mu e^{-i\vec{k}\cdot\vec{r}}\ ,\ a(k')_\nu e^{i\vec{k}'\cdot\vec{r}'} + a(k')^\dagger_\nu e^{-i\vec{k}'\cdot\vec{r}'}] =$$

$$\int \frac{d^3k}{(2\pi)^{3/2}}\frac{d^3k'}{(2\pi)^{3/2}}\beta(k)\alpha(k')\, c_{\mu\nu}\,\delta(\vec{k}-\vec{k}')[e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + e^{i\vec{k}\cdot(\vec{r}'-\vec{r})}] \ .$$

Vi ønsker at dette skal bli lik $-i\eta_{\nu\mu}\delta(\vec{r}-\vec{r}')$, og må derfor forlange at $2\alpha(k)\beta(k)c_{\mu\nu} = -i\eta_{\mu\nu}$. Med normeringen $c_{33} = 1$ har dette løsninger

$$\alpha(k)\beta(k) = \frac{i}{2} \quad , \quad c_{\mu\nu} = -\eta_{\mu\nu}\ .$$

ii) Vi setter dette i uttrykket for Hamiltonfunksjonen

$$-\frac{1}{2}\int d^3r\,\Pi_\mu\Pi^\mu = -\frac{1}{2}\int d^3r \int \frac{d^3k}{(2\pi)^{3/2}}\frac{d^3k'}{(2\pi)^{3/2}}\beta(k)\beta(k')\times$$

$$[a(k)_\mu e^{i\vec{k}\cdot\vec{r}} - a(k)^\dagger_\mu e^{-i\vec{k}\cdot\vec{r}}][a(k')^\mu e^{i\vec{k}'\cdot\vec{r}} - a(k')^{\dagger\mu} e^{-i\vec{k}'\cdot\vec{r}}] =$$

$$\frac{1}{2}\int d^3k\,(\beta(k)^2[a(k)^\dagger a(k) + a(k)a(k)^\dagger] -$$

$$\beta(k)\beta(-k)[a(k)a(-k) + a(k)^\dagger a(-k)^\dagger])\ ,$$

$$-\frac{1}{2}\int d^3r\,\vec{\nabla}A_\mu\cdot\vec{\nabla}A^\mu = -\frac{1}{2}\int d^3r\int\frac{d^3k}{(2\pi)^{3/2}}\frac{d^3k'}{(2\pi)^{3/2}}-\vec{k}\cdot\vec{k}'\,\alpha(k)\alpha(k')\times$$

$$[a(k)_\mu e^{i\vec{k}\cdot\vec{r}}-a(k)_\mu^\dagger e^{-i\vec{k}\cdot\vec{r}}][a(k')^\mu e^{i\vec{k}'\cdot\vec{r}}-a(k')^{\dagger\mu}e^{-i\vec{k}'\cdot\vec{r}}] =$$

$$-\frac{1}{2}\int d^3k\,\{k^2\alpha(k)^2[a(k)^\dagger a(k)+a(k)a(k)^\dagger]+$$

$$k^2\alpha(k)\alpha(k')[a(k)a(-k)+a(k)^\dagger a(-k)^\dagger]\}\ .$$

Vi adderer og forlanger at leddene som går som $a(k)a(-k)+a(k)^\dagger a(-k)^\dagger$ skal forsvinne. Dette er oppfylt dersom $\beta(k)\beta(-k)+k^2\alpha(k)\alpha(-k)=0$. Samholdt med betingelsen $\alpha(k)\beta(k)=i/2$ finner vi løsninger

$$\beta(k)=i\sqrt{\omega_k/2}\qquad\text{og}\qquad\alpha(k)=\sqrt{1/2\omega_k}$$

der $\omega_k^2=\vec{k}^2$. Hamiltonfunksjonen blir da

$$H_0=-\frac{1}{2}\int d^3k\,\omega_k[a(k)_\mu^\dagger a(k)^\mu+a(k)_\mu a(k)^{\dagger\mu}]\ .$$

Kommentar: En enklere metode er å bruke $\Pi_\mu=-\dot{A}_\mu$ til å finne at $\beta(k)=i\omega_k\alpha(k)$. Dispersjonsrelsjonen $\omega_k^2=\vec{k}^2$ kan så finnes fra Euler-Lagrange ligningen $\partial_\nu\partial^\nu A_\mu=0$.

iii) Vi får $H=H_0+\Delta H$, der

$$\Delta H=\int d^3r\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega_k}}[a(k)_\mu\,e^{i\vec{k}\cdot\vec{r}}+a(k)_\mu^\dagger\,e^{-i\vec{k}\cdot\vec{r}}]j^\mu(r)=$$

$$\int\frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{1}{2\omega_k}}[a(k)_\mu J^\mu(k)+a(k)_\mu^\dagger J^\mu(-k)]$$

iv) Vi kan skrive

$$H=-\frac{1}{2}\int d^3k\,\omega_k\times\{[a(k)_\mu^\dagger+\gamma(k)_\mu^*][a(k)^\mu+\gamma(k)^\mu]+$$

$$[a(k)_\mu+\gamma(k)_\mu][a(k)^{\dagger\mu}+\gamma(k)^{*\mu}]-2\gamma_\mu^*(k)\gamma^\mu(k)\}$$

der $\gamma^\mu(k)=J^\mu(k)/(2\pi\omega_k)^{3/2}\sqrt{2}$. Dette er allerede på diagonal form hvis vi innfører nye kreasjons- og annihilasjonsoperatorer:

$$b(k)_\mu^\dagger=a(k)_\mu^\dagger+\gamma(k)_\mu^*\quad,\quad b(k)_\mu=a(k)_\mu+\gamma(k)_\mu\ .$$

Siden $\gamma(k)^\mu$ er et c-tall vil disse oppfylle de samme kommuteringsreglene som de gamle operatorene. Grunntilstandsenergien blir

$$E_0(j)=E_\infty+\int d^3k\,\omega_k\gamma_\mu^*(k)\gamma^\mu(k)=E_\infty+\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\frac{J_\mu(k)J^\mu(-k)}{k^2}$$

der E_∞ er en uendelig normalordningskonstant som ikke avhenger av j^μ. Dersom vi dropper denne og transformerer til det relle rom blir grunntilstandsenergien

$$E_0 = \frac{1}{2}\int d^3r\, d^3r' \, \frac{j^0(\vec{r})j^0(\vec{r}') - \vec{j}(\vec{r})\cdot\vec{j}(\vec{r}')}{4\pi\,|\,\vec{r}-\vec{r}'\,|} ,$$

som er identisk med det klassiske uttrykket for vekselvirkningsenergi mellom ladninger og strømmer.