

Klassisk feldteori: 12.12.87

Læringer

$$1a: \int \sum_a \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} \delta \psi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta (\partial_\mu \psi_a) \right) dx = \int \sum_a \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \right) \delta \psi_a dx + \int \sum_a \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} \delta \psi_a d\sigma = 0$$

overflade
0 på overflaten

Gyldig for vilkårlige $\delta \psi_a$ hver

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}_a} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_a)} = 0 \quad \text{for hver } a = 1, 2, \dots$$

b) \mathcal{L}_m være skalar ved romrotasjon og inversjon

$$\frac{\partial \psi}{\partial x^i} \frac{\partial \phi}{\partial x^{i'}} = a^{i'}_k a^{i''}_l \frac{\partial \phi}{\partial x^k} \frac{\partial \psi}{\partial x^l} = \frac{\partial \psi}{\partial x^k} \frac{\partial \phi}{\partial x^k} \quad \text{da } a^{i'}, a^{i''}_l = \delta_{kk}$$

$$\frac{\partial \phi}{\partial t'} \frac{\partial \psi}{\partial t'} = \frac{\partial \psi}{\partial t} \frac{\partial \phi}{\partial t} \quad \frac{\partial \phi}{\partial t'} = \frac{\partial \phi}{\partial t}$$

$$L_i \frac{\partial \phi}{\partial x^{i'}} = L_i a^{i'}_k \frac{\partial \phi}{\partial x^k} \quad \text{ikke invariant}$$

\mathcal{L}_0 , \mathcal{L}_1 er skalare, men \mathcal{L}_2 ikke e. brukbar

$$c) \mathcal{L}_1 = -\frac{g_0}{2} \left((\nabla \phi)^2 - \frac{1}{c_e^2} \left(\frac{\partial \phi}{\partial t} \right)^2 \right) + K \frac{\partial \phi}{\partial t}$$

$$\frac{\partial \mathcal{L}_1}{\partial \phi} = 0 \quad \frac{\partial \mathcal{L}_1}{\partial (\partial_i \phi)} = -g_0 \partial_i \phi \quad \frac{\partial \mathcal{L}_1}{\partial \dot{\phi}} = g_0 \frac{\dot{\phi}}{c_e^2} + K$$

Feltlikning:

$$-\frac{\partial}{\partial x^i} \left(-g_0 \frac{\partial \phi}{\partial x^i} \right) - \frac{\partial}{\partial t} \left(\frac{g_0}{c_e^2} \frac{\partial \phi}{\partial t} + K \right) = g_0 \frac{\partial^2 \phi}{\partial x^{i^2}} - \frac{g_0}{c_e^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

$$\nabla^2 \phi - \frac{1}{c_e^2} \frac{\partial^2 \phi}{\partial t^2} = 0$$

$$d) x^\mu = (c_e t, x, y, z) \quad \mu = 0, 1, 2, 3 \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^\mu} = -\frac{\partial}{\partial x_\mu} \quad \frac{\partial}{\partial t} = c_e \frac{\partial}{\partial x^0} = c_e \frac{\partial}{\partial x_0}$$

$$x_\mu = (c_e t, -x, -y, -z)$$

$$\mathcal{L}_0 = \frac{g_0}{2} \left(-\frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^i} + \frac{1}{c_e^2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial t} \right) = \frac{g_0}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu}$$

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1d fort.

$$\frac{d\mathcal{L}}{dx^r} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \frac{\partial \varphi}{\partial x^r} + \frac{\partial \mathcal{L}}{\partial (\partial_r \varphi)} \frac{\partial^2 \varphi}{\partial x^r \partial x^r} + \frac{\partial \mathcal{L}}{\partial x^r}$$

$$- \frac{\partial \mathcal{L}}{\partial x^r} = \frac{d}{dx^r} \left(\frac{\partial \mathcal{L}}{\partial (\partial_r \varphi)} \right) \frac{\partial \varphi}{\partial x^r} + \frac{\partial \mathcal{L}}{\partial (\partial_r \varphi)} \frac{\partial^2 \varphi}{\partial x^r \partial x^r} - \frac{d \mathcal{L}}{dx^r} = \frac{d}{dx^r} \left(\frac{\partial \mathcal{L}}{\partial (\partial_r \varphi)} \partial_r p - \delta_r^v \mathcal{L} \right)$$

Hvis $\frac{\partial \mathcal{L}}{\partial x^r} = 0$ fremst vil $\frac{\partial T^v}{\partial x^r} = 0$ med $T^v_r = \frac{\partial \mathcal{L}}{\partial (\partial_r \varphi)} \partial_r p - \delta_r^v \mathcal{L}$

\mathcal{L}_1 avhenger ikke eksplisit av x^r for $r = 0, 1, 2, 3$

$$e) a) g^0 = T^{00} = T^0_0 = \frac{\partial \mathcal{L}_1}{\partial (\partial_0 \varphi)} \partial_0 \varphi - \mathcal{L}_1 = \frac{\partial \mathcal{L}_1}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L}_1 \quad \text{da } \partial_0 p = \frac{1}{c_e} \dot{\varphi}$$

$$= \left(\frac{g_0}{c_e^2} \dot{\varphi} + K \right) \dot{\varphi} + \frac{g_0}{2} (\nabla \varphi)^2 - \frac{g_0}{2 c_e^2} \dot{\varphi}^2 - K \dot{\varphi} = \frac{g_0}{2} \left((\nabla \varphi)^2 + \frac{1}{c_e^2} \dot{\varphi}^2 \right)$$

$$S^i; \xi_i T^{i0} = \xi_i T^i_0 = \xi_i \frac{\partial \mathcal{L}_1}{\partial (\partial_i \varphi)} \partial_i \varphi = -g_0 \partial_i \varphi \cdot \dot{\varphi} = -g_0 \dot{\varphi} \nabla \varphi$$

$$g^i = \frac{1}{c_e} T^{0i} = -\frac{1}{c_e} T^0_i = -\frac{1}{c_e} \frac{\partial \mathcal{L}_1}{\partial (\partial_i \varphi)} \cdot \partial_i \varphi = -\left(g_0 \frac{\dot{\varphi}}{c_e} + K \right) \partial_i \varphi = -\left(\frac{g_0}{c_e} \dot{\varphi} + K \right) \nabla \varphi$$

b) $g_0 \dot{\varphi} = -c_e^2 (p - p_0) \quad \nabla \varphi = \vec{v} \quad g_0 \dot{\varphi} = -(p - p_0)$

$$\mathcal{H} = \frac{1}{2} g_0 v^2 + \frac{1}{2} g_0 \left(\frac{p - p_0}{g_0} \right)^2 c_e^2 = \frac{1}{2} g_0 v^2 + \frac{1}{2 g_0 c_e^2} (p - p_0)^2$$

$$\vec{s} = \frac{c_e^2 (p - p_0) \vec{v}}{} = \frac{(p - p_0) \vec{v}}{}$$

$$\vec{g} = \frac{(p - p_0 - K) \vec{v}}{}$$

Hela luftmassen på stedet går med \vec{v} så rimelig $\vec{g} = p \vec{v}$ dvs $K = -p_0$

$$\sigma^{ik} = -T^{ik} = T^i_k = \frac{\partial \mathcal{L}_1}{\partial (\partial_i \varphi)} \partial_k \varphi - \delta_{ik}^v \mathcal{L}_1 = -g_0 \partial_i \varphi \cdot \partial_k \varphi + \delta_{ik}^v \frac{g_0}{2} [(\nabla \varphi)^2 - \frac{1}{c_e^2} \dot{\varphi}^2] - \delta_{ik}^v K \dot{\varphi}$$

Symmetri i indekrene $\sigma^{ik} = \sigma^{ki}$ så romlig dreiemoment bevarat.

$$T: 1. orden er
- $\sigma^{ik} = -\delta_{ik}^v K \dot{\varphi} + \delta_{ik}^v g_0 \dot{\varphi} = -(p - p_0) \delta_{ik}^v + O(v^2, (p - p_0)^2)$$$

II a)

$$\begin{aligned}\delta(A^\kappa A_\kappa) &= \delta(A^\kappa g_{\mu\lambda} A^\lambda) = \delta A^\kappa g_{\mu\lambda} A^\lambda + A^\kappa \delta g_{\mu\lambda} A^\lambda + A^\kappa g_{\mu\lambda} \delta A^\lambda \\ &= -\Gamma_{\mu\nu}^\kappa A^\mu \delta x^\nu g_{\lambda\lambda} A^\lambda + A^\kappa \frac{\partial g_{\mu\lambda}}{\partial x^\nu} \delta x^\nu A^\lambda - \Gamma_{\nu\mu}^\kappa A^\nu \delta x^\mu g_{\lambda\lambda} A^\lambda = 0 \\ &= -\left(\Gamma_{\mu\lambda}^\kappa g_{\nu\lambda} + \Gamma_{\nu\lambda}^\kappa g_{\mu\lambda} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) A^\kappa A^\nu \delta x^\lambda = 0\end{aligned}$$

Også for alle vektorer og parallelle forskyninger når

$$\Gamma_{\mu\lambda}^\kappa g_{\nu\lambda} + \Gamma_{\nu\lambda}^\kappa g_{\mu\lambda} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = 0$$

b) Skifter vi $\mu \rightarrow \nu \rightarrow \lambda \rightarrow \mu$ i næste nistre likning ovenfor ser vi at også

$$\Gamma_{\nu\mu}^\kappa g_{\lambda\lambda} + \Gamma_{\lambda\mu}^\kappa g_{\nu\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\mu} = 0$$

$$\Gamma_{\lambda\nu}^\kappa g_{\mu\mu} + \Gamma_{\mu\nu}^\kappa g_{\lambda\lambda} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = 0$$

Trekker vi likningene: a) fra sammen av disse to fås

$$(\Gamma_{\mu\nu}^\kappa + \Gamma_{\nu\mu}^\kappa) g_{\lambda\lambda} = \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right)$$

$$\Gamma_{\mu\nu}^\kappa = \frac{1}{2} g^{\lambda\lambda} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right)$$

Har bemerket at $\Gamma_{\mu\nu}^\kappa = \Gamma_{\nu\mu}^\kappa$ (Følger fra antakelsen i g)

om parallelle forskyninger når en krysser $dA^\kappa = dx^\kappa$

$$d(dx^\kappa) = -\Gamma_{\mu\lambda}^\kappa dx^\mu dx^\lambda = -\Gamma_{\lambda\mu}^\kappa dx^\lambda dx^\mu$$

c) Setter $A^\kappa = u^\kappa$ og forskyver langt ut $dx^\lambda = u^\lambda d\tau$

$$\delta u^\kappa = -\Gamma_{\mu\lambda}^\kappa u^\mu u^\lambda d\tau \Rightarrow \frac{du^\kappa}{d\tau} = -\Gamma_{\mu\lambda}^\kappa u^\mu u^\lambda$$

d) På en 2dim. kuleflate

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \mu, \nu = 1, 2$$

$$g_{\mu\nu} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix} \quad g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda \Rightarrow g^{\mu\nu} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2 \theta} \end{pmatrix}$$

Bare $g_{11} \neq 0$ og $g_{22} \neq 0$ og de avhenger bare av $x^1 = \theta$

$$\text{stikk at bare ledd } \frac{\partial g_{22}}{\partial x^1} = 2R^2 \sin \theta \cos \theta \neq 0$$

12.12. F7

4d) fort.

Altså bare $\Gamma'_{22}, \Gamma^2_{12}, \Gamma^2_{21} \neq 0$

$$\Gamma'_{22} = \frac{1}{2} g^{11} \left(-\frac{\partial g_{22}}{\partial x^1} \right) = -\frac{R^2}{R^2 \sin \vartheta} \cos \vartheta = -\frac{\sin \vartheta \cos \vartheta}{\sin^2 \vartheta}$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} \frac{\partial g_{21}}{\partial x^1} = \frac{1}{2} \frac{1}{R^2 \sin^2 \vartheta} 2 R^2 \sin \vartheta \cos \vartheta = \frac{\cos \vartheta}{\sin^2 \vartheta} = \Gamma^2_{21}$$

Gir for parallell forskyvning på horisontal:

$$\delta A^2 = \delta A^1 = -\Gamma'_{22} A^2 dx^2 = -\Gamma'_{22} A^2 d\varphi = 0 \text{ her da } \delta \varphi = 0$$

$$\delta A^1 = \delta A^2 = -\Gamma^2_{12} A^1 \delta x^2 - \Gamma^2_{21} A^2 \delta x^1 = -\Gamma^2_{21} A^2 \delta \vartheta$$

$$\text{Första likning: } \text{S.t. } dA^2 = 0 \quad A^2 = \text{kant} = A_0^2$$

$$\text{Annars gir: } dA^2 = -\frac{\sin \vartheta}{\sin^2 \vartheta} A^2 d\vartheta$$

$$\text{eller } \frac{dA^2}{A^2} = -\frac{\sin \vartheta}{\sin^2 \vartheta} d\vartheta$$

Integgerar från $\vartheta_0 = \frac{\pi}{2}$ till ϑ

$$\ln A^2(\vartheta) - \ln A_0^2 = -(\ln \sin \vartheta - \ln \sin \vartheta_0)$$

$$\frac{A^2(\vartheta)}{A_0^2} = \frac{\sin \vartheta_0}{\sin \vartheta} \quad A^2(\vartheta) = \frac{A_0^2}{\sin^2 \vartheta}$$

Längden blir uforändrat:

$$A^\mu A_\mu = g_{\mu\nu} A^\mu A^\nu = g_{11} A^1 A^1 + g_{22} A^2 A^2 = R^2 (A_0^2)^2 + R^2 \sin^2 \vartheta \frac{(A^2)^2}{\sin^2 \vartheta}$$

$$= R^2 ((A_0^2)^2 + (A_0^2)^2) = \underline{R^2 (A_0^2)^2} = \underline{\overline{A_0^2}}$$