Exam in course no 71562 Critical phenomena
(fag 71562 Kritiske fenomener)
Friday June 6, 1975
9 a.m. - 4 p.m.

1.

Consider a ferromagnetic spin system in an external magnetic field $\mathcal{H}$. The Curie temperature (critical temperature) is T_c and the critical field is $\mathcal{H}_c = 0$.

a) Give definitions of the critical indices $\alpha, \alpha', \beta, \gamma, \gamma'$ and δ for heat capacity, spontaneous magnetization, susceptibility and the critical isotherm.

b) On the basis of a certain simplifying assumption the Landau theory predicts definite values for the critical indices. What is the basic assumption of the Landau theory and what are the resulting values for α, β, γ and δ? (No derivation is required here).

c) The basic assumption of the Landau theory is not fulfilled for the present ferromagnet. In the neighbourhood of the Curie point the singular part of the free energy per spin, f_s, is a homogeneous function of the following form

$$f_s(t,\mathcal{H}) = \begin{cases} t^a \, g_+(\mathcal{H}/t^{\Delta}) & \text{for} \quad t > 0 \\ (-t)^a \, g_-(\mathcal{H}/(-t)^{\Delta}) & \text{for} \quad t < 0 \,, \end{cases} \qquad (1)$$

where $t = (T-T_c)/T_c$. The quantities a and Δ are known constants.

Under the necessary assumption on the functions $g_{\pm}(x)$, deduce the critical indices α, α', β and γ, γ'.

Show that the scaling law

$$\alpha + 2\beta + \gamma = 2 \qquad (2)$$

is fulfilled, irrespective of the values of a and Δ.

d) The singular part of the free energy per spin satisfies a functional equation of the following form

$$f_s(\lambda_t t, \lambda_h \mathcal{H}) = A\, f_s(t, \mathcal{H}) \quad . \tag{3}$$

For a given value of the constant A, relate the constants λ_t and λ_h to the previous constants a and Δ. Show that

$$\alpha = 2 - \frac{\ln A}{\ln \lambda_t} \quad . \tag{4}$$

2.

In the Wilson theory of critical phenomena one studies a transformation of the physical system described by interaction parameters $K_1, K_2, \ldots$ onto a new system of the same nature, described, however, by a different set of interaction parameters $K_1', K_2', \ldots$. The transformation implies a reduction of the number of degrees of freedom by some factor, s, say. Give examples of such Wilson transformations.

One presupposes that the connection between the two sets of coupling constants is known,

$$K_n' = \chi_n(K_1, K_2, \ldots) \quad . \qquad (n = 1, 2, \ldots) \tag{5}$$

Explain briefly how one in principle goes about in order to deduce values of critical indices from knowledge of the transformation equations (5). Make the assumptions you consider necessary.

3

For Wilson's d-dimensional spin-field model, in zero magnetic field characterized by two interaction parameters u and v, one can concoct a transformation whose main ingredient consists of integrating out the short-wavelength components of the spin field. The transformation has the following properties:

i) The number of degrees of freedom is reduced by a factor $s = 2$.

ii) The interaction parameters in the transformed system (u',v') can be calculated under the assumption $0 \leq u << 1$, with the result

$$v' = 4v + 12(1+v)^{-1} u - 36(1+v)^{-3} u^2 + \mathcal{O}(u^3) \; .$$
$$u' = 16 \cdot 2^{-d} u - 144 \cdot 2^{-d} (1+v)^{-2} u^2 + \mathcal{O}(u^3) \; . \qquad (6)$$

a) Explain that the transformation in the neighbourhood of dimensionality $d = 4$ has two fixed points,

$$F_1 : \; v^* = u^* = 0$$
$$F_2 : \; v^* = -4u^* = -\epsilon \tfrac{4}{9} \ln 2 + \mathcal{O}(\epsilon^2) \; ,$$

with $\epsilon = 4-d$. Which fixed point is the important one for $d < 4$, and why?

b) Show that linearization of the equations (6) around the fixed point F_2 yields to first order in the small quantity ϵ :

$$v'-v^* = (4-\epsilon\tfrac{4}{3} \ln 2)(v-v^*) + (12-\epsilon\tfrac{8}{3} \ln 2)(u-u^*)$$
$$u'-u^* = (1-\epsilon \ln 2)(u-u^*) \; .$$

What are the eigenvalues of the linearized transformation?

c) Calculate for $d < 4$ the specific heat critical index α to first order in ϵ .