Løsningsskisse

Oppgave 1

a. & b. som forel.

c. Problemet for $q>0$ er oscillatorproblemet, med grensebetingelsene $\psi(0) = \psi(\infty) = 0$. De oscillator-egenfunksjonene som tilfredsstiller dette kravet er derfor egenf., og det er $\psi_{2n+1}^{\text{oscillator}}$ som er antisymmetriske:

$$E_m = \left(2n+\tfrac{3}{2}\right)\hbar\omega \; ; \; \varphi_m = \sqrt{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{e^{-m\omega q^2/2\hbar}}{\sqrt{2^{2n+1}(2n+1)!}} \; H_{2n+1}\left(q\sqrt{m\omega/\hbar}\right),$$

$\sqrt{2}$-faktor pga normalisering i $(0,\infty)$.

d. Grunntilstanden er

$$\varphi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-m\omega q^2/2\hbar} \, 2q\sqrt{m\omega/\hbar}$$

Utvikling i egenfunksjonene for det symmetriske potensial gir

$$c_m = \int dq \; \psi_m^*(q)\, \varphi_0(q)$$

$$c_0 = \int_0^\infty dq \sqrt{\frac{m\omega}{\pi\hbar}} \; e^{-m\omega q^2/\hbar} \cdot 2q\sqrt{m\omega/\hbar}$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty dx\, x\, e^{-x^2} = \underline{\frac{1}{\sqrt{\pi}}}$$

$$c_1 = \int_0^\infty dq \sqrt{\frac{m\omega}{\pi\hbar}} \; e^{-m\omega q^2/\hbar} \cdot 2q\sqrt{m\omega/\hbar} \cdot q\sqrt{2m\omega/\hbar}$$

$$= \frac{2\sqrt{2}}{\sqrt{\pi}}\int_0^\infty x^2 e^{-x^2}\, dx = \underline{\frac{1}{\sqrt{2}}},$$

så $\underline{\underline{P_0 = |c_0|^2 = \frac{1}{\pi} \; ; \; P_1 = |c_1|^2 = \frac{1}{2}}}$

e. Dette var ved $t=0$. Den tidsavhengige bølgefunksjon er

$$\psi(\vec{r},t) = \sum_n c_n \, \varphi_n(\vec{r}) \, e^{i\omega t(n+\frac{1}{2})}$$

Skal vise at

$$\psi(\vec{r},T_N) = \psi(\vec{r},0) \quad \text{for}$$

$$T_N = 4\pi N/\omega .$$

Dette følger umiddelbart av

$$e^{i\omega T_N (n+\frac{1}{2})} = e^{2\pi i \, N(2n+1)} = 1$$

f.

$$\psi(q,0) = \frac{1}{\sqrt{2}} \varphi_0(q) + \frac{1}{\sqrt{2}} \varphi_1(q)$$

så

$$\langle E \rangle = \frac{1}{2} \frac{\hbar\omega}{2} + \frac{1}{2} \frac{3\hbar\omega}{2} = \underline{\underline{\hbar\omega}}$$

Bølgefunksjonen ved $t>0$

$$\psi(q,t) = \frac{1}{\sqrt{2}} \varphi_0(q) e^{-\frac{i}{\hbar}E_0 t} + \frac{1}{\sqrt{2}} \varphi_1(q) e^{-\frac{i}{\hbar}E_1 t}.$$

$$\langle q \rangle = \int_{-\infty}^{+\infty} |\psi|^2 q \, dq = \frac{1}{2}\int_{-\infty}^{+\infty} dq \, q \, \varphi_0(q)\,\varphi_1(q) \left(e^{-\frac{i}{\hbar}(E_1-E_0)t} + c.c.\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cdot \cos\omega t \; \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dx \; x \cdot \frac{2x}{\sqrt{2}} e^{-x^2} = \underline{\underline{\cos\omega t \sqrt{\frac{\hbar}{2m\omega}}}}$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \psi^*(q,t) \frac{\hbar}{i}\frac{\partial}{\partial q} \psi(q,t) \, dq =$$

$$= \frac{1}{2}\frac{\hbar}{i}\int_{-\infty}^{+\infty} \left(\varphi_0 \, e^{+\frac{i}{\hbar}E_0 t} + \varphi_1 \, e^{+\frac{i}{\hbar}E_1 t}\right)\left(\varphi_0' \, e^{-\frac{i}{\hbar}E_0 t} + \varphi_1' \, e^{-\frac{i}{\hbar}E_1 t}\right) dq$$

$$= \frac{1}{2}\frac{\hbar}{i}\int_{-\infty}^{+\infty} dq \left[\varphi_1\varphi_0' \, e^{+\frac{i}{\hbar}(E_1-E_0)t} + \varphi_0\varphi_1' \, e^{-\frac{i}{\hbar}(E_1-E_0)t}\right]$$

da $\int \varphi_0\varphi_0' \, dq$ og $\int \varphi_1 \varphi_1' \, dq$ begge er 0.

Da $\int \psi_0 \psi_1' \, dq = -\int \psi_0' \psi_1 \, dq$ ved delvis integrasjon og $E_1 - E_0 = \hbar\omega$ fås

$$\langle p \rangle = \hbar \sin\omega t \cdot \int_{-\infty}^{+\infty} \psi_1 \psi_0' \, dq =$$

$$= \hbar \sin\omega t \cdot \sqrt{\frac{m\omega}{\pi\hbar}} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} 2x \, e^{-\frac{x^2}{2}} \left(-x e^{-\frac{x^2}{2}}\right) dx =$$

$$= -\sqrt{\frac{m\omega\hbar}{2}} \sin\omega t$$

[Kunne alternativt benyttet Ehrenfests teorem + verdiene $\langle p \rangle_0 = 0$; $\langle q \rangle_0 = \sqrt{\frac{\hbar}{2m\omega}}$].

Oppgave 2

a) Som komp.

b) $\psi_n(x) = \sqrt{\frac{2}{L}} \sin(n\pi x/L)$, $\quad E_n^0 = \frac{n^2\pi^2\hbar^2}{2m L^2}$; $n = 1, 2, 3, \ldots$

c) Første ordens korreksjon til E_n^0:

$$\langle n | V | n \rangle = \int_0^L \psi_n^2(x) \, \lambda x^2 \, dx = \frac{2\lambda}{L} \int_0^L \sin^2 \frac{n\pi x}{L} \, x^2 \, dx$$

$$= \lambda L^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

Oppgave 3

a) Som forel.

b) $$\frac{d}{dt}\langle \vec{p} \rangle = \frac{i}{\hbar}\langle [mgz, \vec{p}\,] \rangle = \frac{i}{\hbar} mg \left(-\frac{\hbar}{i} \vec{e}_z\right) = -mg\,\vec{e}_z$$

$$\langle \vec{p} \rangle_t = \langle \vec{p} \rangle_0 - mgt\,\vec{e}_z$$

$$\frac{d}{dt}\langle \vec{r} \rangle = \frac{i}{\hbar}\left\langle \left[\frac{\vec{p}^{\,2}}{2m}, \vec{r}\right]\right\rangle = \frac{i}{2m\hbar}\sum_{k=x,y,z}[p_k^2, k]$$

Da $[p_x^2, x] = p_x(p_x x - x p_x) + (p_x x - x p_x) p_x = \frac{\hbar}{i} 2p_x$, fås

$\frac{d}{dt}\langle \vec{r} \rangle = \frac{1}{m}\langle \vec{p} \rangle = \frac{1}{m}\langle \vec{p} \rangle_0 - mgt\,\vec{e}_z$. Integrasjon gir

$$\langle \vec{r} \rangle_t = \langle \vec{r} \rangle_0 + \frac{t}{m}\langle \vec{p} \rangle_0 - \frac{1}{2} g t^2 \vec{e}_z$$