Kont eks 13 5 1983 Innf i kvantemek.

I a) $H\Psi(\vec r,t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec r)\right]\Psi(\vec r,t) = i\hbar\frac{\partial\Psi}{\partial t}$

Stasjonær $H\psi(\vec r) = E\psi(\vec r)$ $\quad \Psi(\vec r,t) = \psi(\vec r)\, e^{-\frac{i}{\hbar}Et}$

Middelverdi i tilstand $\Psi(\vec r,t)$ $\quad \langle F\rangle = \int \Psi^*(\vec r,t)\, F_{op}(\vec r, -i\hbar\nabla, t)\,\Psi(\vec r,t)\, d^3r$

b)
$$\frac{d}{dt}\int \Psi^* F \Psi\, d^3r = \int\left(\frac{\partial\Psi^*}{\partial t}F\Psi + \Psi^* F\frac{\partial\Psi}{\partial t} + \Psi^*\frac{\partial F}{\partial t}\Psi\right)d^3r$$
$$= \int\left[\frac{1}{-i\hbar}\Psi^* H F\Psi + \frac{1}{i\hbar}\Psi^* F H\Psi + \Psi^*\frac{\partial F}{\partial t}\Psi\right]d^3r = \underline{\frac{i}{\hbar}\langle[H,F]\rangle + \left\langle\frac{\partial F}{\partial t}\right\rangle}$$

c)
$$\frac{d}{dt}\langle\vec p\rangle = \frac{i}{\hbar}\left\langle\left[\frac{\vec p^2}{2m} + mgz, \vec p\right]\right\rangle = \frac{i}{\hbar}mg\langle[z, p_z\hat e_z]\rangle = \frac{i}{\hbar}mg\,\hat e_z\, i\hbar = -mg\,\hat e_z$$
$$\underline{\langle\vec p\rangle_t = \langle\vec p\rangle_0 - mgt\,\hat e_z}$$
$$\frac{d}{dt}\langle\vec r\rangle = \frac{i}{\hbar}\left\langle\left[\frac{\vec p^2}{2m}, \vec r\right]\right\rangle = \frac{i}{2m\hbar}\left\langle\left[\Sigma p_k^2, \Sigma x_j\hat e_j\right]\right\rangle$$
$$= \frac{i}{2m\hbar}\left\langle\frac{\hbar}{i}2\vec p\right\rangle \quad \text{da} \quad [p_x^2, x] = p_x(p_x x - x p_x) + (p_x x - x p_x)p_x = \frac{\hbar}{i}2p_x$$
$$= \frac{1}{m}\langle\vec p\rangle = \frac{1}{m}\langle\vec p\rangle_0 - gt\,\hat e_z$$
$$\underline{\langle\vec r\rangle_t = \langle\vec r\rangle_0 + \frac{t}{m}\langle\vec p\rangle_0 - \frac{1}{2}gt^2\hat e_z}$$

II a) Hermitesk hvis A oppfyller $\int\psi^* A\varphi\, d^3r = \int(A\psi)^*\varphi\, d^3r$ for alle tillatte tilstandsfunksjoner ψ, φ.

Vi sier at $A = A^+$ (Definisjon på adjungert operator A^+ til operatoren A: $\int\psi^* A^+\varphi\, d^3r = \int(A\psi)^*\varphi\, d^3r$)

Operatoren $A \equiv F + F^+$ er hermitesk
$$\int\psi^* A^+\varphi\, d^3r = \int\left((F+F^+)\psi\right)^*\varphi\, d^3r = \int\psi^* F^+\varphi\, d^3r + \int\psi^*(F^+)^+\varphi\, d^3r$$
$$= \int\psi^*(F^+ + F)\varphi\, d^3r \qquad \text{fordi} \quad (F^+)^+ = F$$

Altså $\underline{A^+ = F^+ + F = F + F^+ = A}$

Tilsvarende for $B \equiv i(F - F^+)$
$$\int\psi^* B^+\varphi\, d^3r = \int\left(i(F - F^+)\psi\right)^*\varphi\, d^3r = -i\int\psi^*\left(F^+ - (F^+)^+\right)\varphi\, d^3r$$
$$\underline{B^+ = -i(F^+ - F) = i(F - F^+) = B}$$

II b) $\int \psi_a^* \hat{A} \psi_a \, d\vec{r} = a \int \psi_a^* \psi_a \, d\vec{r} = a$ Når $\hat{A}$ er hermitesk er dette lik $a = a^*$ og

$\int (\hat{A}\psi_a)^* \psi_a \, d\vec{r} = a^* \int \psi_a^* \psi_a \, d\vec{r} = a^*$ $\underline{a \text{ er reell.}}$

Variansen i målevediene:

$$(\Delta A)^2 = \int \psi_a^* (\hat{A} - \langle A \rangle)^2 \psi_a \, d\vec{r} = \int \psi_a^* \hat{A}^2 \psi_a \, d\vec{r} - \langle A \rangle^2 \int \psi_a^* \psi_a \, d\vec{r}$$

$$= \int (\hat{A}\psi_a)^* \hat{A}\psi_a \, d\vec{r}^3 - a^2 = a^* a - a^2 = \underline{\underline{0}}$$

c)

$$\langle F \rangle = \int \left(\sum c_n^* \psi_n^*\right) F \left(\sum c_k \psi_k\right) d\vec{r} = \sum_{n,k} c_n^* c_k \int \psi_n^* F \psi_k \, d\vec{r} = \sum_{n,k} c_n^* c_k f_k \delta_{nk}$$

$$= \underline{\sum_n |c_n|^2 f_n}$$

$|c_n|^2$ gir sannsynligheten for å finne egenverdien f_n når en måler F i tilstanden ψ.

III a)

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{V_0}{\cosh^2 \frac{x}{a}}\right) \psi = E \psi$$

Ny variabel:

$q = \sinh^2 \frac{x}{a}$ $(q \geq 0)$ $\cosh^2 \frac{x}{a} = 1 + \sinh^2 \frac{x}{a} = 1 + q$

$\cosh \frac{x}{a} = \sqrt{1+q}$

$\sinh \frac{x}{a} = \pm \sqrt{q}$ for $x \gtrless 0$ $q = 0$ for $x = 0$, $q \to \infty$ for $x \to \pm\infty$

$$\frac{dq}{dx} = \frac{2}{a} \sinh \frac{x}{a} \cosh \frac{x}{a} = \pm \frac{2}{a} \sqrt{q(1+q)}$$

$$\frac{d^2 q}{dx^2} = \frac{2}{a^2}\left(\cosh^2 \frac{x}{a} + \sinh^2 \frac{x}{a}\right) = \frac{2}{a^2}(1 + 2q)$$

$$\frac{d}{dx} = \frac{dq}{dx} \frac{d}{dq} = \pm \frac{2}{a} \sqrt{q(1+q)} \frac{d}{dq}$$

$$\frac{d^2}{dx^2} = \left(\frac{dq}{dx}\right)^2 \frac{d^2}{dq^2} + \frac{d^2 q}{dx^2} \frac{d}{dq} = \frac{4}{a^2} q(1+q) \frac{d^2}{dq^2} + \frac{2}{a^2}(1 + 2q) \frac{d}{dq}$$

$$\left[-\frac{\hbar^2 4}{2ma^2}\left(q(1+q)\frac{d^2}{dq^2} + \left(\frac{1}{2} + q\right)\frac{d}{dq}\right) - E - \frac{V_0}{1+q}\right] \psi = 0$$

$$\left[q(1+q) \frac{d^2}{dq^2} + \left(\frac{1}{2} + q\right) \frac{d}{dq} + \frac{1}{4} \frac{2ma^2}{\hbar^2} E + \frac{1}{4} \frac{\frac{2ma^2 V_0}{\hbar^2}}{1+q}\right] \psi(q) = 0$$

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III b) $\psi = (1+q)^\lambda u$ $\qquad \psi' = \lambda(1+q)^{\lambda-1}u + (1+q)^\lambda u'$

$$\psi'' = \lambda(\lambda-1)(1+q)^{\lambda-2}u + 2\lambda(1+q)^{\lambda-1}u' + (1+q)^\lambda u''$$

Innsatt

$$(1+q)^\lambda \left[q(1+q)u'' + 2\lambda q u' + \lambda(\lambda-1)\frac{q}{1+q}u + (\tfrac12+q)u' + \frac{\frac12+q}{1+q}\lambda u + \frac{\varepsilon}{4}u + \frac{\frac14\nu}{1+q}u\right] = 0$$

Ordnet

$$q(1+q)u'' + \left((2\lambda+1)q+\tfrac12\right)u' + \left(\lambda^2+\tfrac14\varepsilon\right)u + \frac{-\lambda(\lambda-1)-\frac12\lambda+\frac14\nu}{1+q}u = 0$$

Velger λ slik at siste ledd forsvinner: $\lambda^2 - \frac12\lambda = \frac14\nu$

$$\lambda = \tfrac14\left(1\pm\sqrt{4\nu+1}\right)$$

Velger $-\sqrt{\ }$ på grunn av konvergens av normeringsintegralet for $q\to\infty$

$$\underline{\lambda = -\frac14\left(\sqrt{\frac{8ma^2V_0}{\hbar^2}+1}-1\right)}$$

c) Rekkeutvikler $\qquad u = \sum_{\mu=0}^{\infty} c_\mu q^{s+\mu}$

Innsatt

$$\sum_\mu \left\{ q(1+q)(s+\mu)(s+\mu-1)c_\mu q^{s+\mu-2} + \left((2\lambda+1)q+\tfrac12\right)(s+\mu)c_\mu q^{s+\mu-1} + \left(\lambda^2+\tfrac14\varepsilon\right)c_\mu q^{s+\mu}\right\} = 0$$

$$q^{s-1}\left\{s(s-1)c_0 + \tfrac12 s c_0\right\} + \sum_{\mu=0} q^{s+\mu}\left\{(s+\mu+1)(s+\mu)c_{\mu+1} + (s+\mu)(s+\mu-1)c_\mu + (2\lambda+1)(s+\mu)c_\mu + \tfrac12(s+\mu+1)c_{\mu+1} + \left(\lambda^2+\tfrac14\varepsilon\right)c_\mu\right\} = 0$$

Koeff. for q^{s-1} gir $\left(s(s-1)+\frac12 s\right)c_0 = 0$ gir $\underline{s = 0 \text{ eller } \frac12}$

og for $q^{s+\mu}$ gir rekursjonsformelen

$$c_{\mu+1} = -\frac{(s+\mu)(s+\mu+2\lambda)+\left(\lambda^2+\frac14\varepsilon\right)}{(s+\mu+1)(s+\mu+\frac12)}c_\mu$$

For $\mu\to\infty$ $\frac{c_{\mu+1}}{c_\mu}\to -1$. Skal normeringsintegralet konvergere må rekken bryte av for $\mu_{max} = k$.

Normeringsintegralet blir:

$$\int_{-\infty}^{\infty}|\psi|^2dx = 2\int_0^\infty (1+q)^{2\lambda}|u|^2\frac{dq}{\left|\frac{dq}{dx}\right|} = 2\int_0^\infty \frac{(1+q)^{2\lambda}}{\frac2a\sqrt{q(1+q)}}|u|^2dq = a\int_0^\infty\frac{(1+q)^{2\lambda-\frac12}}{q^{1/2}}|u|^2dq$$

Konvergens ved $q\to0$ i orden både for $s=0$ og $s=\frac12$.

For $q\to\infty$ må $q^{2\lambda-1}q^{2(s+k)} < q^{-1}$ $\qquad 2\lambda+2(s+k)<0$ $\qquad \underline{k<-\lambda-s}$

(λ må velges negativ som vi har gjort.)

Må altså ha avbrudd for $\mu_{max}=k<-\lambda-s$:

$(s+k)(s+k+2\lambda)+\left(\lambda^2+\frac14\varepsilon\right)=0$ med $k=0,1,\dots k_{max}<-\lambda-s$

$$\varepsilon = -4(\lambda+k+s)^2 = -(2\lambda+n)^2 \quad\text{med}\quad n = 2k+2s = \begin{cases}0,2,4,\dots n_{max} & \text{med } s=0\\ 1,3,5,\dots n_{max} & \text{med } s=\frac12\end{cases}$$

Altså:

$$\underline{E_n = -\frac{\hbar^2}{2ma^2}(-2\lambda-n)^2 = -\frac{\hbar^2}{2ma^2}\left(\frac12\sqrt{\frac{8ma^2V_0}{\hbar^2}+1}-n-\frac12\right)^2}$$

$\underline{n_{max} < -2\lambda = \frac12\sqrt{\frac{8ma^2V_0}{\hbar^2}}-\frac12}$

$\underline{n=0,1,2,3,\dots n_{max}}$