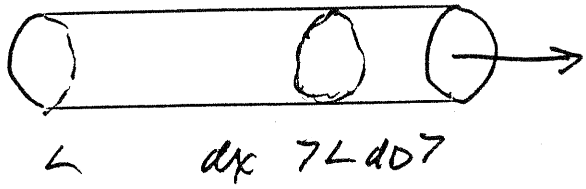
Oppgave 1

a)



Spenningen, kraft pr. flateenhet, er proporsjonal med strekket, relativ lengdeforandring $\tau = E\frac{dD}{dx}$

$$\frac{dD}{dx} = D_m k\cos(kx-\omega t) \quad \text{konst. } t$$

$$\Rightarrow \tau = ED_m k\cos(kx-\omega t)$$

Største forandring $\tau_m = ED_m k$

$$k = \frac{\omega}{v} \wedge v = \sqrt{\frac{E}{\rho}} \Rightarrow k = \omega\sqrt{\frac{\rho}{E}}$$

$$\tau_m = ED_m\omega\sqrt{\frac{\rho}{E}} = \sqrt{E\rho}\,D_m\omega$$

$$\tau_m = \sqrt{2{,}0\cdot10^{11}\cdot8{,}0\cdot10^{3}}\cdot1{,}0\cdot10^{-4}\cdot2\pi\cdot1000\ \mathrm{N/m^2}$$

$$\underline{\tau_m = 2{,}51\cdot10^{7}\ \mathrm{N/m^2}}$$

b) Intensitet

$$I = -E\left(\frac{\partial D}{\partial x}\right)\left(\frac{\partial D}{\partial t}\right)$$

$$\frac{\partial D}{\partial t} = -D_m\omega\cos(kx-\omega t)$$

$$I = -ED_m k\cos(kx-\omega t)(-D_m)\omega\cos(kx-\omega t)$$

$$k = \frac{\omega}{v}$$

$$I = E D_m^2 \frac{\omega^2}{v} \cos^2(kx - \omega t)$$

$$v = \sqrt{\frac{E}{\rho}}$$

$$I = \sqrt{E\rho}\, \omega^2 D_m^2 \cos^2(kx - \omega t)$$

$$\overline{I} = \sqrt{E\rho}\, \omega^2 D_m^2\, \overline{\cos^2(kx - \omega t)}$$

$$\overline{\cos^2(kx - \omega t)} = \frac{1}{2}$$

$$\overline{I} = \frac{1}{2}\sqrt{E\rho}\, \omega^2 D_m^2 = \frac{1}{2}\sqrt{2{,}0\cdot 10^{11} \cdot 8{,}0\cdot 10^{3}} \cdot 4\pi^2 \cdot 10^6 \cdot 1{,}0\cdot 10^{-8} \,\frac{W}{m^2}$$

$$\underline{\overline{I} = 7{,}89\cdot 10^6\, W/m^2}$$

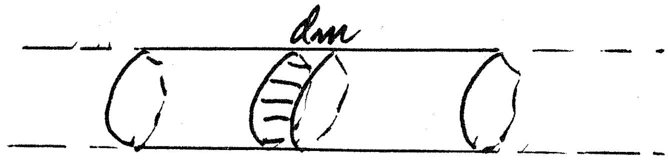
c) Midlere energitetthet

$$\overline{w} = \frac{\overline{I}}{v} = \frac{1}{2}\frac{E D_m^2 \omega^2}{v^2} = \frac{1}{2}\rho\,\omega^2 D_m^2$$

$$\overline{w} = \frac{1}{2} \cdot 8{,}0\cdot 10^3 \cdot 4\pi^2 \cdot 10^6 \cdot 1{,}0\cdot 10^{-8}\, J/m^3$$

$$\underline{\overline{w} = 1{,}58\cdot 10^3\, J/m^3}$$

d)



Partikkel (element) hastighet $\left(\frac{\partial D}{\partial t}\right)$

Elementets kinetiske energi

$$dE_k = \frac{1}{2}\,dm\left(\frac{\partial D}{\partial t}\right)^2$$

tetthet $\rho = \frac{dm}{dV} \Rightarrow dm = \rho\, dV$

der dV er volumet av elementet

$$dE_k = \frac{1}{2}\rho\, dV \left(\frac{\partial D}{\partial t}\right)^2$$

Kinetisk energitetthet

$$w_k = \frac{dE_k}{dV} = \frac{1}{2}\rho\left(\frac{\partial D}{\partial t}\right)^2 = \frac{1}{2}\rho D_m^2\omega^2\cos^2(kx-\omega t)$$

midlere energitetthet

$$\overline{w_k} = \frac{1}{2}\rho D_m^2\omega^2\,\overline{\cos^2(kx-\omega t)}$$

$$\underline{\overline{w_k} = \frac{1}{4}\rho\omega^2 D_m^2}$$

Da den totale energitetthet

$$\overline{w} = \frac{1}{2}\rho\,\omega^2 D_m^2$$

er $\quad \overline{w_k} = \frac{1}{2}\overline{w}$

Den totale energitettheten må ha bidrag både fra kinetisk og potensiell energi

$$\overline{w} = \overline{w_k} + \overline{w_p}$$

der $\overline{w_p}$ er midlere energitetthet som skyldes potensiell energi

$$\Rightarrow \quad \underline{\overline{w_p} = \overline{w_k} = \frac{1}{4}\rho\omega^2 D_m^2}$$

Oppgave 2

a) *i* Gasslign. $pV = nRT$

for $n = 1$ mol er volumet

$$V = \frac{RT}{p} = \frac{kN_A T}{p}$$

Ved standard betingelser

$$V = \frac{1{,}38\cdot 10^{-23}\cdot 6{,}02\cdot 10^{23}\cdot 273}{1{,}013\cdot 10^{5}}\,m^3 = 22{,}4\cdot 10^{-3}\,m^3$$

Antall luftmolekyler pr. vol.-enhet (partikkel-tettheten

$$\underline{n_V} = \frac{N_A}{V} = \frac{6{,}02\cdot 10^{23}}{22{,}4\cdot 10^{-3}}\,m^{-3} = \underline{2{,}69\cdot 10^{25}/m^3}$$

ii Middel fri vei

$$l_m = \frac{1}{\sqrt{2}\,\pi n_V d^2} = \frac{1}{\sqrt{2}\,\pi\cdot 2{,}69\cdot 10^{25}\cdot 9\cdot 10^{-20}}\,m$$

d molekylets diameter

$$\underline{l_m = 9{,}30\cdot 10^{-8}\,m}$$

iii

1. Middelverdien av fri vei, middel fri vei

$$l_m = \frac{\int_0^\infty x\,p(x)\,dx}{\int_0^\infty p(x)\,dx} = \frac{\int_0^\infty x e^{-\alpha x}\,dx}{\int_0^\infty e^{-\alpha x}\,dx}$$

$$\underline{l_m} = \frac{\frac{1}{\alpha^2}}{\frac{1}{\alpha}} = \underline{\frac{1}{\alpha}}$$

2. Brøkdel fri vei $x < l_m$

$$\int_0^{l_m} p(x)\,dx = \int_0^{l_m} \frac{1}{l_m} e^{-\frac{x}{l_m}} dx = 1 - e^{-1} = \underline{0{,}63}$$

Brøkdel fri vei $x > l_m$

$$\int_{l_m}^{\infty} p(x)\,dx = \int_{l_m}^{\infty} \frac{1}{l_m} e^{-\frac{x}{l_m}} dx = e^{-1} = \underline{0{,}37}$$

b) i Diffusjonskonstant

$$D = \frac{1}{3} \bar{v}\, l_m$$

midlere hastighet

$$\bar{v} = 1{,}60 \sqrt{\frac{kT}{m}} = 1{,}60 \sqrt{\frac{1{,}38 \cdot 10^{-23} \cdot 273}{29 \cdot 1{,}66 \cdot 10^{-27}}}\ \text{m/s}$$

$$\bar{v} = 1{,}60 \cdot 280\ \text{m/s} = 448\ \text{m/s}$$

$$\underline{D} = \frac{1}{3} \cdot 448 \cdot 9{,}30 \cdot 10^{-8}\ \text{m}^2/\text{s} = \underline{1{,}39 \cdot 10^{-5}\ \frac{\text{m}^2}{\text{s}}}$$

ii Partikkelstrømtettheten

$$J = -D \frac{\partial n_V}{\partial x}$$

Tilnærmet, numerisk

$$J = D \frac{\Delta n_V}{\Delta x}$$

der $\Delta n_V = \frac{1}{2} n_V = \frac{1}{2} \cdot 2{,}69 \cdot 10^{25}$ molekyler/m^3

$$\Delta x = L = 2{,}0 \cdot 10^{-3}\ \text{m}$$

$$\underline{J} = 1{,}39 \cdot 10^{-5} \cdot \frac{2 \cdot 1{,}64 \cdot 10}{2{,}0 \cdot 10^{-3}} \frac{\text{molekyler}}{s} = \underline{9{,}35 \cdot 10^{22} \frac{\text{molekyler}}{s}}$$

iii) Partikkelstrømtettheten

$$J = \left[\frac{\#}{m^2 s}\right] = \frac{\#}{A \cdot t}$$

Midlere diffusjonstid for et molekyl

$$\underline{t} = \frac{1}{A J} = \frac{1}{2{,}0 \cdot 10^{-9} \cdot 9{,}35 \cdot 10^{22}} s = \underline{5{,}35 \cdot 10^{-15} s}$$

c) Ved konstant trykk

$$D = c\,T^{3/2} \qquad D_1 = c\,T_1^{3/2}$$

$$\Rightarrow \quad D_1 = D \left(\frac{T_1}{T}\right)^{3/2} = \left(\frac{293}{273}\right)^{3/2} \cdot 1{,}39 \cdot 10^{-5} \frac{m^2}{s}$$

$$\underline{D_1} = 1{,}11 \cdot 1{,}39 \cdot 10^{-5} \frac{m^2}{s} = \underline{1{,}54 \cdot 10^{-5} \frac{m^2}{s}}$$

Oppgave 3

a) $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi$$

$$\psi=Ax\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$\frac{d\psi}{dx}=Ax\left(-\frac{m\omega 2x}{2\hbar}\right)\exp\left(-\frac{m\omega x^2}{2\hbar}\right)+A\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$\frac{d^2\psi}{dx^2}=\left(\frac{m^2\omega^2x^2}{\hbar^2}-\frac{3m\omega}{\hbar}\right)Ax\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

Innsatt i Schrödingerlign.

$$-\frac{\hbar^2}{2m}\left(\frac{m^2\omega^2x^2}{\hbar^2}-\frac{3m\omega}{\hbar}\right)Ax\exp\left(-\frac{m\omega x^2}{2\hbar}\right)+$$

$$+\frac{1}{2}m\omega^2x^2Ax\exp\left(-\frac{m\omega x^2}{2\hbar}\right)=EAx\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{m^2\omega^2x^2}{\hbar^2}-\frac{3m\omega}{\hbar}\right)+\frac{1}{2}m\omega^2x^2=E$$

ordnet

$$x^2\left(\frac{1}{2}m\omega^2-\frac{1}{2}m\omega^2\right)+\frac{3}{2}\hbar\omega=E$$

uttrykket i parentesen er null

$$\Rightarrow \psi(x)=Ax\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

er egenfunksjon

med egenverdi $\underline{E=\frac{3}{2}\hbar\omega}$

b) Normering $\int_{-\infty}^{+\infty} |\Psi(x)|^2 dx = 1$

kaller $a^2 = \frac{m\omega}{\hbar} \Rightarrow |\Psi(x)|^2 = A^2 x^2 e^{-a^2x^2}$

$$A^2 \int_{-\infty}^{+\infty} x^2 e^{-a^2x^2} dx = 1$$

$$A^2 \cdot \frac{1}{2}\sqrt{\frac{\pi}{a^6}} = 1$$

$$\Rightarrow \underline{A = 2a\sqrt{\frac{a}{2\sqrt{\pi}}}}$$

$$\Psi(x) = 2a\sqrt{\frac{a}{2\sqrt{\pi}}}\, x \exp\left(-\frac{a^2x^2}{2}\right)$$

c) Av symmetriskegrunner $\underline{\langle x \rangle = 0}$

Kan også regnes ut

$$\langle x \rangle = \int_{-\infty}^{+\infty} dx \cdot x \left(Axe^{-\frac{a^2x^2}{2}}\right)^2 = 0$$

Ubestemthet Δx

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle \text{ da } \langle x \rangle = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} 2a\sqrt{\frac{a}{2\sqrt{\pi}}}\, xe^{-\frac{a^2x^2}{2}} \cdot x^2 \cdot 2a\sqrt{\frac{a}{2\sqrt{\pi}}}\, xe^{-\frac{a^2x^2}{2}} dx$$

$$= 4a^2 \cdot \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} x^4 e^{-a^2x^2} dx$$

$$\langle x^2\rangle = \frac{2a^3}{\sqrt{\pi}}\cdot\frac{3}{4}\sqrt{\frac{\pi}{a^{10}}} = \frac{3}{2a^2}$$

$$\underline{\Delta x} = \sqrt{\langle x^2\rangle} = \underline{\frac{1}{2a}\sqrt{6}} = \underline{\sqrt{\frac{3}{2}\cdot\frac{\hbar}{m\omega}}}$$

d) Forventningsverdien av p:

$$\underline{\langle p\rangle = 0}$$

da partikkelen beveger seg frem og tilbake med samme numeriske verdi av p.

Kan også regnes ut

$$\langle p\rangle = \int_{-\infty}^{+\infty} Axe^{-\frac{a^2x^2}{2}}\left(-i\hbar\frac{d}{dx}\right)Axe^{-\frac{a^2x^2}{2}}dx = 0$$

Ubestemthet Δp

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \langle p^2\rangle \text{ da } \langle p\rangle = 0$$

$$\langle p^2\rangle = \int_{-\infty}^{+\infty} 2a\sqrt{\frac{a}{2\sqrt{\pi}}}\,xe^{-\frac{a^2x^2}{2}}\left(-\hbar^2\frac{d^2}{dx^2}\right)2a\sqrt{\frac{a}{2\sqrt{\pi}}}\,xe^{-\frac{a^2x^2}{2}}dx$$

$$\langle p^2\rangle = 4a^2\frac{a}{2\sqrt{\pi}}(-\hbar^2)\int_{-\infty}^{+\infty} xe^{-\frac{a^2x^2}{2}}\left(\frac{d^2}{dx^2}\right)xe^{-\frac{a^2x^2}{2}}dx$$

Fra a)

$$\left(\frac{d^2}{dx^2}\right)xe^{-\frac{a^2x^2}{2}} = -3a^2xe^{-\frac{a^2x^2}{2}} + a^4x^3e^{-\frac{a^2x^2}{2}}$$

$$\Rightarrow \langle p^2\rangle = -\frac{4a^3\hbar^2}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\left(-3a^2x^2e^{-a^2x^2} + a^4x^4e^{-a^2x^2}\right)dx$$

$$\int_{-\infty}^{+\infty} x^2 e^{-a^2x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^6}}$$

$$\int_{-\infty}^{+\infty} x^4 e^{-a^2x^2}\,dx = \frac{3}{4}\sqrt{\frac{\pi}{a^{10}}}$$

$$\Rightarrow \langle p^2\rangle = -\frac{4a^3\hbar^2}{2\sqrt{\pi}}\left[-3a^2\cdot\frac{1}{2}\sqrt{\frac{\pi}{a^6}} + a^4\cdot\frac{3}{4}\sqrt{\frac{\pi}{a^{10}}}\right]$$

$$\langle p^2\rangle = \frac{3}{2}a^2\hbar^2$$

$$\Delta p = \sqrt{\langle p^2\rangle} = \sqrt{\frac{3}{2}}\,a\hbar$$

idet $a^2 = \frac{m\omega}{\hbar}$

$$\underline{\Delta p} = \underline{\sqrt{\frac{3}{2}\hbar m\omega}}$$

e) Etter c) og d)

$$\Delta x = \sqrt{\frac{3}{2}}\,\frac{1}{a} \qquad \Delta p = \frac{3}{2}a\hbar$$

$$\underline{\Delta x\cdot\Delta p = \frac{3}{2}\hbar}$$

i overensstemmelse med Heisenbergs ubestemthetsrelasjon $(\Delta x\cdot\Delta p \geq \frac{1}{2}\hbar)$

f) Forventningsverdien av kinetisk energi

$$\langle E_k\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{2m}\cdot\frac{3}{2}a^2\hbar^2 = \frac{1}{2m}\cdot\frac{3}{2}\frac{m\omega}{\hbar}\hbar^2$$

etter d)

$$\underline{\langle E_k\rangle} = \underline{\frac{1}{2}\cdot\frac{3}{2}\hbar\omega}$$

Forventningsverdien av potensiell energi

$$\underline{\langle U\rangle} = E - \langle E_k\rangle = \frac{3}{2}\hbar\omega - \frac{1}{2}\cdot\frac{3}{2}\hbar\omega = \underline{\frac{1}{2}\cdot\frac{3}{2}\hbar\omega}$$

eller $\langle U\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$