Løsningsforslag til våreksamen 1995 i fag 74327: Relativistisk kvantemekanikk

Oppgave 1

a) Skriver ut (2) på komponentform:

$$\frac{\partial}{\partial t} E_k = \epsilon_{ijk} \frac{\partial}{\partial x_i} B_j$$

Altså:

$$i \frac{\partial}{\partial t} E_k = -1 \cdot i \epsilon_{ijk} \frac{\partial}{\partial x_i} iB_j$$

$$= -i\,(-i)\,\epsilon_{ikj} \frac{\partial}{\partial x_i} iB_j \qquad (A)$$

fortegnsskifte p.g.a. permutering av indeksene.

Bruker vi (7) nå, kan vi skrive (A) som:

$$i \frac{\partial}{\partial t} E_k = -i (S_i)_{kj} \frac{\partial}{\partial x_i} iB_j \qquad (B)$$

På vektorform blir dette (5).

Omformingen av (4) til (6) er helt analog.

b)

$$\underline{[S_i, S_j]_{mn}} = -\{\epsilon_{imk}\,\epsilon_{jkn} - \epsilon_{jmk}\,\epsilon_{ikn}\}$$

$$= \epsilon_{kim}\,\epsilon_{kjn} - \epsilon_{kjm}\,\epsilon_{kin}$$

$$= \cancel{\delta_{ij}\,\delta_{mn}} - \delta_{in}\,\delta_{jm} - \cancel{\delta_{ji}\,\delta_{mn}} + \delta_{jn}\,\delta_{mi}$$

$$= \delta_{jn}\,\delta_{im} - \delta_{jm}\,\delta_{in} = \epsilon_{kji}\;\epsilon_{knm}$$

$$= -\epsilon_{ijk}\;\epsilon_{knm} = \epsilon_{ijk}\,\epsilon_{kmn}$$

$$= i\,\epsilon_{ijk}\,(-i)\,\epsilon_{kmn}$$

$$= \underline{i\,\epsilon_{ijk}\,(S_k)_{mn}} \qquad (C)$$

c) Skriver vi ut (11) på 2×2 form, får vi

$$i\,\frac{\partial}{\partial t}\begin{Bmatrix} \vec{E} \\ i\vec{B} \end{Bmatrix} = -i\begin{Bmatrix} 0 & \vec{S} \\ \vec{S} & 0 \end{Bmatrix}\begin{Bmatrix} \vec{E} \\ i\vec{B} \end{Bmatrix} \qquad (D)$$

som på 2-komponent form gir (5) og (6).

d) Vi "kvadrerer" (11):

$$-\frac{\partial^2}{\partial t^2}\psi = -i\vec{\alpha}\cdot\vec{\nabla}\, i\frac{\partial}{\partial t}\psi = -(\vec{\alpha}\cdot\vec{\nabla})(\vec{\alpha}\cdot\vec{\nabla})\psi$$

$$\partial_i \equiv \frac{\partial}{\partial x_i}$$

$$= -\alpha_i\alpha_j\,\partial_i\partial_j\psi$$

Siden $\partial_i\partial_j = \partial_j\partial_i$

$$= -\frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)\,\partial_i\partial_j\psi$$

$$= -\frac{1}{2}\{\alpha_i, \alpha_j\}\,\partial_i\partial_j\psi$$

Altså: $\underline{\frac{\partial^2}{\partial t^2}\psi = \frac{1}{2}\{\alpha_i, \alpha_j\}\,\partial_i\partial_j\psi}$ (E)

Maxwells ligninger

$$\vec{\nabla}\times \;\Big|\; \frac{\partial\vec{E}}{\partial t} = \vec{\nabla}\times\vec{B}$$

$$\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{B}}_{=0 \text{ p.g.a. (3)}}) - \nabla^2\vec{B}$$

Bruker (4)

$$-\frac{\partial^2}{\partial t^2}\vec{B} = -\nabla^2\vec{B} \;\rightarrow\; \underline{\frac{\partial^2}{\partial t^2}i\vec{B} = \nabla^2 i\vec{B}} \quad (F)$$

På samme måte:

$$\vec{\nabla}\times \Big| \quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

$$\frac{\partial}{\partial t}\nabla\times\vec{B} = -\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0 \text{ p.g.a.}}) + \nabla^2\vec{E}$$

ligning (2)

$$\underline{\frac{\partial^2}{\partial t^2}\vec{E} = \nabla^2\vec{E}} \qquad (G)$$

Skriver (F) og (G) ved hjelp av ψ, definert i (10), har man:

$$\underline{\frac{\partial^2}{\partial t^2}\psi = \nabla^2\psi} \qquad (H)$$

Sammenligner man (E) og (H) får man at

$$\underline{\underline{\{\alpha_i, \alpha_j\} = 2\delta_{ij}}} \qquad (I)$$

e) En ligning av formen $i\gamma^\mu\partial_\mu\psi = 0$ er kun kovariant hvis den etter en Lorentz transformasjon ser ut som $i\gamma^\mu\partial'_\mu\psi' = 0$

γ-matrisene er de <u>samme</u> som før transformasjon

f) Vi antar at (14) er kovariant. Da har vi:

$$i\gamma^\mu \frac{\partial}{\partial x'^\mu} \psi'(x') = 0 \quad \rightarrow$$

$$i\gamma^\mu \underbrace{\frac{\partial x^\nu}{\partial x'^\mu}} \frac{\partial}{\partial x^\nu} L(\Lambda)\psi(x) = 0$$

$$x^\mu = x^\mu(x') = \frac{\partial x^\mu}{\partial x'^\nu} x'^\nu = (\Lambda^{-1})^\mu{}_\nu x'^\nu$$

$$L(\Lambda)^{-1} \,|\quad i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu L(\Lambda) \frac{\partial}{\partial x^\nu} \psi(x) = 0$$

$$i \underbrace{L(\Lambda)^{-1}\gamma^\mu L(\Lambda) (\Lambda^{-1})^\nu{}_\mu} \frac{\partial}{\partial x^\nu} \psi(x) = 0$$

hvis kovariant: $= \gamma^\nu$

$$\rightarrow \quad \underline{L(\Lambda)^{-1}\gamma^\mu L(\Lambda) (\Lambda^{-1})^\nu{}_\mu = \gamma^\nu}$$

som er ensiret (15).

g) (14) er <u>ikke</u> kovariant. Velg $\mu = 0$ i (15).
Da har vi ved bruk av (12):

$$L(\Lambda)\gamma^0 L(\Lambda)^{-1} = L(\Lambda)L(\Lambda)^{-1} = \underline{1 = (\Lambda^{-1})^0{}_\nu \gamma^\nu}$$

som er selvmotsigende.

Oppgave 2

a)

Hvis $C^{-1} = i\gamma^0\gamma^2$, så må vi ha at

$$(i\gamma^2\gamma^0)(i\gamma^0\gamma^2) = -\gamma^2\underbrace{\gamma^0\gamma^0}_{\{\gamma^0,\gamma^0\} = 2g^{00} = 2}\gamma^2 = -\underbrace{\gamma^2\gamma^2}_{\{\gamma^2,\gamma^2\} = 2g^{22} = -2} = -(-1) = 1$$

b)

$$\begin{aligned}
\underline{C\gamma^\mu C^{-1}} &= i\gamma^2\gamma^0\gamma^\mu i\gamma^0\gamma^2 \\
&= -\gamma^2\gamma^0\gamma^\mu\gamma^0\gamma^2 \\
&= -\gamma^2\gamma^0\{2g^{\mu 0} - \gamma^0\gamma^\mu\}\gamma^2 \\
&= -2g^{\mu 0}\gamma^2\gamma^0\gamma^2 + \gamma^2\gamma^0\gamma^0\gamma^\mu\gamma^2 \\
&= 2g^{\mu 0}\gamma^2\gamma^2\gamma^0 + \gamma^2\gamma^\mu\gamma^2 \\
&= -2g^{\mu 0}\gamma^0 + \gamma^2\{2g^{2\mu} - \gamma^2\gamma^\mu\} \\
&= \underline{-2g^{\mu 0}\gamma^0 + 2g^{2\mu}\gamma^2 + \gamma^\mu} \qquad (J)
\end{aligned}$$

$$\left.\begin{aligned}
\mu = 0 &\rightarrow C\gamma^0C^{-1} = -\gamma^0 = -(\gamma^0)^T \\
\mu = 1 &\rightarrow C\gamma^1C^{-1} = +\gamma^1 = -(\gamma^1)^T \\
\mu = 2 &\rightarrow C\gamma^2C^{-1} = -\gamma^2 = -(\gamma^2)^T \\
\mu = 3 &\rightarrow C\gamma^3C^{-1} = +\gamma^3 = -(\gamma^3)^T
\end{aligned}\right\}$$

Som er (17).

(K)

c) Adjungerte Dirac ligning:

$$\bar{\psi}\,[\gamma^{\mu}(-i\partial_{\mu} - eA_{\mu}) - m] = 0$$

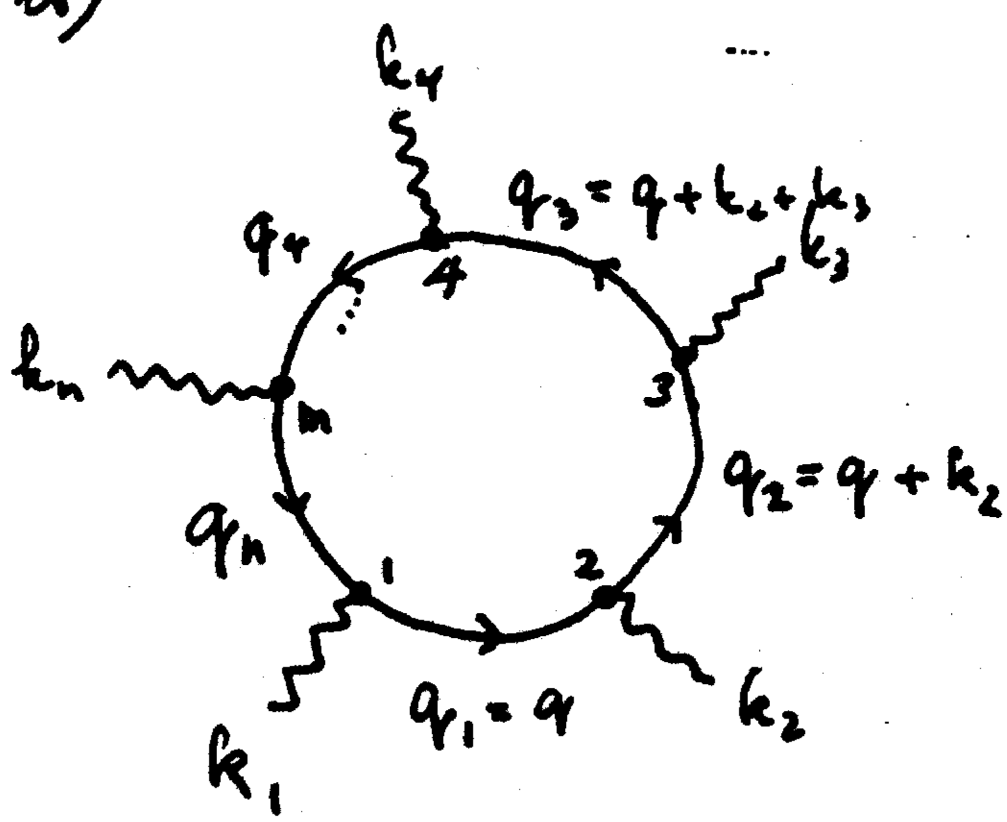
Transponerer:

$$C| \quad [\gamma^{\mu T}(i\partial_{\mu} + eA_{\mu}) + m]\,\bar{\psi}^{T} = 0$$

$$[\underbrace{C\gamma^{\mu T}C^{-1}}_{= -\gamma^{\mu} \text{ på ligning (17)}}(i\partial_{\mu} + eA_{\mu}) + m]\,C\bar{\psi}^{T} = 0$$

$$[\gamma^{\mu}(i\partial_{\mu} + eA_{\mu}) - m]\,C\bar{\psi}^{T} = 0 \qquad (2)$$

d)



Alle momenter k_i er positive i retningen inn til løkken.

Momentet som strømmer mellom knute i og $i+1$:

$$q_i = q \quad \text{hvis } i = 1$$

$$q_i = q + \sum_{j=2}^{i} k_i \quad \text{hvis } i > 1.$$

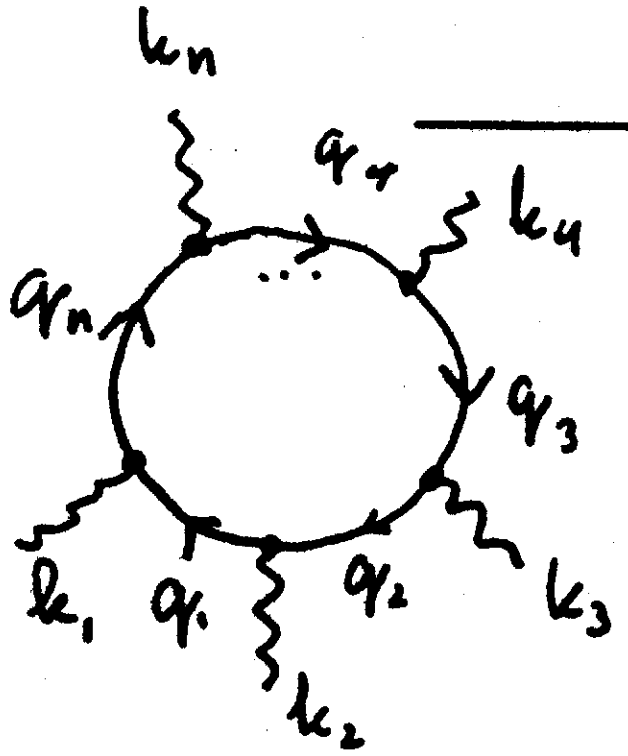
Videre har vi forøvrig

$$\sum_{i=1}^{m} k_i = 0$$

Vi har droppet foton polarisasjonene, siden disse ikke nødvendigvis er ytre linjer...

Da er $\mathcal{M}_a = (-1)\int \frac{dq}{(2\pi)^4}\, tr\big(ie\gamma^{\alpha_1} iS_F(q_1) \cdots$

$$\cdots ie\gamma^{\alpha_n} iS_F(q_n)\big) \qquad (M)$$



definisjon av q_i som tidligere

fortegn på q_i som før.

$$\mathcal{M}_b = (-1)\int \frac{dq}{(2\pi)^4}\, tr\big(ie\gamma^{\alpha_1} iS_F(-q_n) \cdots$$

$$ie\gamma^{\alpha_n} iS_F(-q_1)\big) \qquad (N)$$

e)

Feynman propagatoren: $S_F(q) = \dfrac{\gamma^\mu q_\mu + m}{q^2 - m^2 + i\epsilon}$

$$\underline{C S_F(q) C^{-1}} = \frac{C\gamma^\mu C^{-1} q_\mu + m}{q^2 - m^2 + i\epsilon} = \frac{-\gamma^{\mu T} q_\mu + m}{q^2 - m^2 + i\epsilon}$$

$$= \underline{S_F^T(-q)} \qquad (O)$$

$$\underline{tr(A^T B^T C^T \cdots N^T)} = A^T_{ij} B^T_{jk} C^T_{kl} \cdots N^T_{mi}$$

$$= A_{ji} B_{kj} C_{lk} \cdots N_{im} = N_{im} \cdots C_{lk} B_{kj} A_{ji}$$

$$= \underline{tr(N \cdots CBA)} \qquad (P)$$

$$\underline{\mathcal{M}_b} = (-1)\int \frac{dq}{(2\pi)^4}\, tr\left(ie\gamma^{\alpha_1}\, iS_F(-q_n) \cdots ie\gamma^{\alpha_n}\, iS_F(-q_1)\right)$$

$$= (-1)\int \frac{dq}{(2\pi)^4}\, tr\Big(\underbrace{C^{-1}C}_{=1}\, ie\gamma^{\alpha_1} C^{-1}C\, iS_F(-q_n) C^{-1}C \cdots$$

$$\cdots \underbrace{C^{-1}C}_{=1}\, ie\gamma^{\alpha_n} \underbrace{C^{-1}C}_{=1}\, iS_F(-q_1)\Big)$$

$$= (-1)\int \frac{dq}{(2\pi)^4}\, tr\left(ie(-\gamma^{\alpha_1})^T\, iS_F^{\,T}(q_n) \cdots \right.$$

$$\left. \cdots ie(-\gamma^{\alpha_n})^T\, iS_F^{\,T}(q_1)\right)$$

$$= (-1)^n(-1)\int \frac{dq}{(2\pi)^4}\, tr\left(ie\,\gamma^{\alpha_1, T}\, iS_F^{\,T}(q_n) \right.$$

$$\left. \cdots ie(\gamma^{\alpha_n})^T\, iS_F^{\,T}(q_1)\right)$$

$$= (-1)^n(-1)\int \frac{dq}{(2\pi)^4}\, tr\left(iS_F(q_1)\, ie\gamma^{\alpha_n} \cdots \right.$$

$$\left. \cdots ie\gamma^{\alpha_1}\right)$$

$$= (-1)^n(-1)\int \frac{dq}{(2\pi)^4}\, tr\left(ie\gamma^{\alpha_1}\, iS_F(q_1) \cdots \right.$$

$$\left. \cdots ie\gamma^{\alpha_n}\, iS_F(q_n)\right)$$

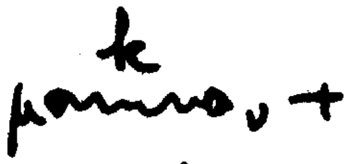
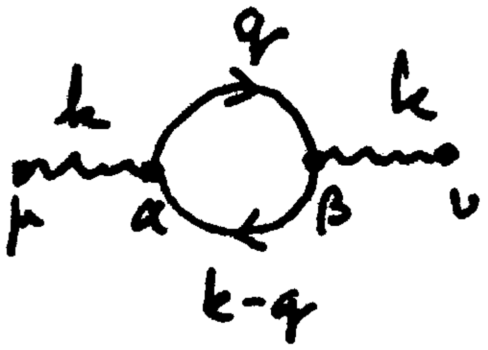
$$\underline{\underline{= (-1)^n \mathcal{M}_a}} \qquad (Q)$$

som skulle vises.

Oppgave 3

a)

 $= iD_F^{\mu\nu}(k) = \dfrac{-ig^{\mu\nu}}{k^2+i\epsilon}$

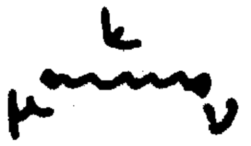
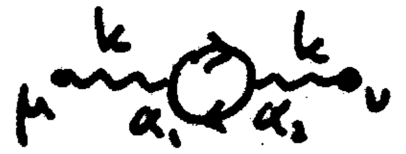
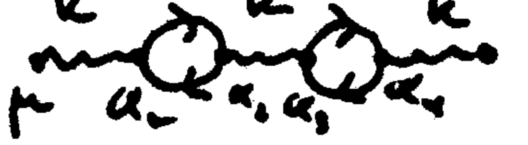
 $+$  $= iD_F'^{\mu\nu}(k)$

$$iD_F'^{\mu\nu}(k) = iD_F^{\mu\nu}(k) + iD_F^{\mu\alpha}(k)(-1)\int\frac{dq}{(2\pi)^4}$$

$$\mathrm{tr}\left(ie\gamma_\alpha\, iS_F(q)\, ie\gamma_\beta\, iS_F(q-k)\right) iD_F^{\beta\nu}(k) \qquad (R)$$

Altså: $$i\Pi_{\alpha\beta}(k) = (-1)\int\frac{dq}{(2\pi)^4}$$

$$\underline{\mathrm{tr}\left(ie\gamma_\alpha\, iS_F(q)\, ie\gamma_\beta\, iS_F(q-k)\right)} \qquad (S)$$

b)  $+$  $+$  $+ \cdots$

$$= \frac{-ig^{\mu\nu}}{k^2+i\epsilon} + \frac{(-ig^{\mu\alpha_1})}{k^2+i\epsilon}\, ia^2 g^{\alpha_1\alpha_2}\cdot \frac{(-ig^{\alpha_2\nu})}{k^2+i\epsilon}$$

$$+\left(\frac{-ig^{\mu\alpha_1}}{k^2+i\epsilon}\right) ia^2 g^{\alpha_1\alpha_2}\left(\frac{-ig^{\alpha_2\alpha_3}}{k^2+i\epsilon}\right) ia^2 g^{\alpha_3\alpha_4}\left(\frac{-ig^{\alpha_4\nu}}{k^2+i\epsilon}\right)+\cdots$$

$$= -ig^{\mu\nu}\Big(\frac{1}{k^2+i\epsilon}+\frac{1}{k^2+i\epsilon}a^2\frac{1}{k^2+i\epsilon}$$

$$+\frac{1}{k^2+i\epsilon}a^2\frac{1}{k^2+i\epsilon}a^2\frac{1}{k^2+i\epsilon}+\cdots\Big)$$

$$= \frac{-ig^{\mu\nu}}{k^2-a^2+i\epsilon} \qquad (T)$$

<u>a opptrer som en masse.</u>

c) $$i\,\Pi_{\alpha\beta}(k) = (-1)\int\frac{d^4q}{(2\pi)^4}\,\mathrm{tr}\left(ie\gamma_\alpha\, iS_F(q)\, ie\gamma_\beta\, iS_F(q-k)\right)$$

$\sim q^3dq$ $\qquad\sim\frac{1}{q}$ $\qquad\sim\frac{1}{(q-k)}\to\frac{1}{q}$ når $k\to 0$

Altså

$$i\,\Pi_{\alpha\beta}(k\to 0)\sim\int\frac{q^3dq}{q\cdot q}\sim\int q\,dq$$

som divergerer.

<u>a^2 er dessverre uendelig!</u>