The Norwegian University of Science and Technology
Department of physics

Contact person during the examination:
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Examination in the course 74 350 Classical field theory
Wednesday November 24, 1999
Time: 09.00–14.00

Allowed aids: (Alternative B): Allowed pocket calculator.
Rottmann, *Mathematische Formelsammlung.*
Barnett and Cronin, *Mathematical Formulae.*
Øgrim og Lian, *Størrelser og enheter i fysikk og teknikk.*
Bronstein, Semendjajew, Musiol, Mühlig, *Taschenbuch der Mathematik.*

Some useful constants:

Newton's gravitational constant:	$G = 6,673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
The speed of light in vacuum:	$c = 299\,792\,458\,\mathrm{m/s}$
The Hubble parameter, present value:	$H_0 = 0,65\ 10^{-10}/\mathrm{years}$

Problem 1:

a) Deduce the equations of motion for a relativistic particle of mass m og electric charge q from the Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\Phi + q\boldsymbol{v}\cdot\boldsymbol{A}\ .$$

Here $\boldsymbol{v}$ is the velocity of the particle, Φ is the electromagnetic scalar potential, and $\boldsymbol{A}$ the (three-dimensional) electromagnetic vector potential.

b) Assume that $\Phi = 0$, and that the magnetic flux density $\boldsymbol{B} = \nabla\times\boldsymbol{A}$ is constant and non-zero both in time and space. Assume e.g. that $\boldsymbol{B}$ points along the z axis.
What symmetries and conservation laws exist then in this system?
Noethers theorem states that if an infinitesimal transformation $\boldsymbol{r}\mapsto\boldsymbol{r}+\Delta\boldsymbol{r}$ gives that

$$L \mapsto L + \frac{\mathrm{d}(\Delta M)}{\mathrm{d}t}\ ,$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\boldsymbol{v}}\cdot\Delta\boldsymbol{r} - \Delta M\right) = 0\ .$$

Problem 2:

Given a contravariant vector field $V^\alpha = V^\alpha(x^0, x^1, x^2, x^3)$, and given the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\eta_{\alpha\beta}\,\eta^{\mu\nu}\,V^\alpha{}_{,\mu}\,V^\beta{}_{,\nu} - V^\rho{}_{,\sigma}\,V^\sigma{}_{,\rho}\right).$$

Here $V^\alpha{}_{,\mu}$ is the partial derivative of V^α, whereas $\eta_{\alpha\beta}$ is the metric tensor of the special theory of relativity,

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

a) Find the Euler–Lagrange equations. How many equations are there?

b) What symmetries and conservation laws are there in this theory?
Give a brief answer, derivations are not required.

c) How can the Lagrangian density be modified so that the theory will be invariant under arbitrary coordinate transformations?

Problem 3:

The point of departure for the cosmological standard model is the Friedmann–Robertson–Walker metric (the FRW metric), where the line element is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a^2\,\mathrm{d}\sigma^2\ .$$

Here t is "universal time", and $a = a(t)$ is a time dependent scale factor, of dimension length. Furthermore, $\mathrm{d}\sigma^2$ is the spatial, three dimensional line element,

$$\mathrm{d}\sigma^2 = \frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\,\sin^2\theta\,\mathrm{d}\varphi^2\ .$$

The radial coordinate r and the polar angles θ and φ are all dimensionless, so that σ is a dimensionless measure of distance. The constant k is either 0, 1 or -1.
By definition, the coordinates r, θ and φ are constant (time independent) for a galaxy following the average expansion of the universe. Here we will neglect the proper motion of the galaxies, which is in addition to the average motion. In this approximation, the dimensionless distance (the coordinate distance) σ between two arbitrary galaxies is constant, while the physical distance $a(t)\sigma$ (measured in kilometer or light year) increases as the universe expands.
The Hubble parameter is defined in the cosmological standard model as

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}\ .$$

The present values of different cosmological variables are usually identified by an index 0. For example, t_0, a_0 and H_0 are the present values of t, a and H.

a) Assume that the light from a galaxy is emitted at the time $t = t_1$, and is observed here at $t = t_2$.
Show that the coordinate distance to this galaxy is

$$\sigma(t_1, t_2) = c \int_{t_1}^{t_2} \frac{\mathrm{d}t}{a(t)} \,. \tag{1}$$

b) If the light from the same galaxy is emitted a little later, at $t = t_1 + \Delta t_1$, then it is observed here at $t = t_2 + \Delta t_2$. Show that we then have

$$\frac{\Delta t_1}{a(t_1)} = \frac{\Delta t_2}{a(t_2)} \,.$$

c) Derive the formula for the cosmological redshift of the spectral lines in the light from a distant galaxy, valid accordig to this model.
That is, express the ratio λ_2/λ_1 in terms of the expansion factors $a(t_1)$ and $a(t_2)$.
We take λ_1 to denote the wave length of the light when it is emitted at $t = t_1$, and λ_2 to denote the wave length of the light when it is observed at $t = t_2$.

d) Assume that

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^n ,$$

where the power n is either $1/2$, for a radiation dominated universe, or $2/3$, for a matter dominated universe. This gives in particular that $a(t) = 0$ at $t = 0$.
Show that $\sigma(t_1, t_2)$, as given in equation (1), then has a finite value in the limit $a(t_1) \to 0$.
Find the limiting value $\sigma(0, t_2)$ (depending on the power n).

e) Take $n = 2/3$ and $t_2 = t_0 =$ "now", use the value of H_0 given in the table on page 1, and find a numerical value for $a_0\,\sigma(0, t_0)$, which is the present distance, in light years, to the most distant galaxies we can observe.
Find also a numerical value for t_0, and compare the two answers. Comments?

f) What does the above result, that $\sigma(t_1, t_2)$ is finite in the limit $a(t_1) \to 0$, mean for the possibility of a physical interaction between two galaxies, or more generally two regions of the universe, early in the history of the universe?
This is a problem (a *causality problem*) for the cosmological standard model. Why?

g) In an inflationary universe, where the vacuum energy dominates, the Hubble parameter H is constant, and the expansion is exponential,

$$a(t) = a_0\, \mathrm{e}^{H\,(t - t_0)} \,.$$

Does the same causality problem exist in a universe which expands exponentially?
Give reasons for your answer.