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EKSAMEN I FAG 74942 RENORMALISERINGSTEORI I STATISTISK FYSIKK

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Hjelpemidler: Lommeregner, Rottmann: Mathematische Formelsammlung.

Problem 1

The effective Landau-Ginzburg-Wilson Hamiltonian for a scalar s^4-model in d spatial dimensions reads

$$\mathcal{H} = \frac{1}{2}\int d^d x \left\{ (\nabla s(\vec{x}))^2 + r s^2(\vec{x}) + 2u s^4(\vec{x}) \right\} \tag{1}$$

with the partition function given symbolically as $Z = \int Ds\ e^{-\mathcal{H}}$.

a. Study the Gaussian model, in which $u=0$ in (1), by a simple rescaling renormalization group (RG) transformation, in which $\vec{x}_L = \vec{x}/L$ and $s_L = cs$. Determine the fixed point and the thermal exponent, y_t. Add a small magnetic field and determine the magnetic exponent, y_h .

b. Use the same rescaling RG to discuss the relevance (in the RG sense) of the us^4-term in (1). Define and determine the upper critical dimensionality.

c. When we use, as we do here, (1) to study the critical properties of uniaxial ferromagnets, would extra terms of the form

$$\int d^d x \left\{ a[\nabla^2 s(\vec{x})]^2 + b[s(\vec{x})\nabla s(\vec{x})]^2 + c[(\nabla s(\vec{x}))^2]^2 \right\}$$

be acceptable from a symmetry point of view? If so, determine their relevance for critical properties.

Problem 2

In Fourier language, the n-component spin model with cubic anisotropy gives an effective Landau-Ginzburg-Wilson Hamiltonian of the form

$$= \frac{1}{2}\int_0^\Lambda d\vec{k}(k^2+r)\sum_{\alpha=1}^{n} s_\alpha(\vec{k})\ s_\alpha(-\vec{k})$$

$$+ u\int_0^\Lambda d\vec{k}_1..d\vec{k}_4 \sum_{\alpha=1}^{n} s_\alpha(\vec{k}_1)\ s_\alpha(\vec{k}_2)\sum_{\beta=1}^{n} s_\beta(\vec{k}_3)s_\beta(\vec{k}_4)\delta(\vec{k}_1+...+\vec{k}_4)$$

$$+ v\int_0^\Lambda d\vec{k}_1..d\vec{k}_4 \sum_{\alpha=1}^{n} s_\alpha(\vec{k}_1)...\ s_\alpha(\vec{k}_4)\delta(\vec{k}_1+...+\vec{k}_4)\ .$$

The RG transformation reads, to leading order in $\epsilon = 4 - d$ and in differential form (with K_4= konst.),

$$\frac{dr}{dL} = 2r + \frac{4K_4}{1+r}[(n+2)u+3v]$$

$$\frac{du}{dL} = \epsilon u - \frac{4K_4}{(1+r)^2}[(n+8)u^2+6uv]$$

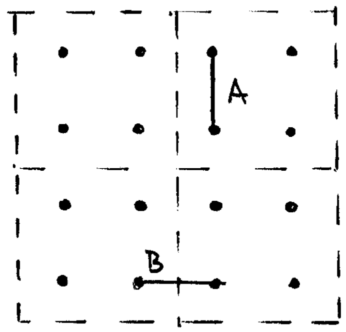
$$\frac{dv}{dL} = \epsilon v - \frac{4K_4}{(1+r)^2}[9v^2+12uv]\ .$$

a. Determine the fixed points of this RG transformation to leading order in ϵ .

b. With a RG transformation involving a finite scaling factor L , the fixed point eigenvalues are written in the form $\lambda_i = L^{y_i}$. Show that when a <u>differential</u> RG , like the one given above, is linearized around a fixed point, the eigenvalues of the corresponding matrix can be directly identified as the exponents y_i .

Problem 3

We shall now study the Ising model on a square lattice by a real space RG transformation. We group the spins of the infinite lattice into cells as shown in the figure.



A: Intracell coupling

B: Intercell coupling

The transformation is defined by the democratic rule, supplemented with the compromise stipulation, i.e.,

RG: The cell spin is up (down) if the majority of spins in the cell is up (down). If two spins in a cell are up and two down, the configuration contributes with half its weight to cell spin up and half its weight to cell spin down.

We write the Hamiltonian (with a factor $-1/k_BT$ absorbed) as

$$H(s) = K \sum_{\langle ij \rangle} + h \sum_i s_i + \text{more complicated terms},$$

where the first sum goes over all nearest neighbor pairs.
For the questions a,b,c and d, we restrict ourselves to the even subspace, i.e., we set equal to zero all terms in $H(s)$ with an odd number of spins.

a. Disregard all couplings except between nearest neighbors, and split the Hamiltonian into an intracell part, H^0, and an intercell part, V (see the figure). Show that to $\mathcal{O}(V)$ the RG transformation defined above becomes

$$K' = 2KF^2(K)$$

where

$$F(K) = \frac{e^{4K} + 2}{e^{4K} + 6 + e^{-4K}} .$$

b. Draw a rough sketch of $K'=K'(K)$ and discuss qualitatively the dynamics of this transformation.

c. Determine the fixed points of the transformation, if necessary, numerically.

d. Calculate the thermal exponent, y_t, to $\mathcal{O}(V)$.

e. Add a small magnetic field and calculate the magnetic exponent, y_h, to $\mathcal{O}(V^0)$.

[For comparison, the exact values (Onsager) are: $K_c = 0.4406\ldots$, $y_t=1$, $y_h = 1.875$].