EKSAMEN I EMNE FY3006 Målesensorer/transdusere

EKSAMEN I EMNE FY8901 Målesensorer/transdusere

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$\boxed{\textit{Model Answers}}$

The last equation in the examination paper was number 16. The first equation in this document has been numbered 17, and likewise for the figures. For marks alloted to separate parts of each problem, see Table 1.

Table 1: Marks alloted to separate parts of each problem

Part(s)	Problem 1	Problem 2	Problem 3
a	13		
b	12		
1–3		3	
4		12	
5		4	
a[1]			15
b			10

[1] 3 marks for each item.

Problem 1

Part a: Let $Q(t)$ denote the charge stored by the capacitor C_s. We take as our starting point the following equations:

$$V_\mathrm{out} = i_1 R_L = \frac{Q}{C_\mathrm{s}}, \tag{17}$$

$$i_2 = \frac{\mathrm{d}Q}{\mathrm{d}t} = C_\mathrm{s}\frac{\mathrm{d}V_\mathrm{out}}{\mathrm{d}t}, \tag{18}$$

$$i_2 = I_A - i_1. \tag{19}$$

Upon equating the right-hand sides of the last two equations, we get:

$$I_A - i_1 = C_\mathrm{s}\frac{\mathrm{d}V_\mathrm{out}}{\mathrm{d}t}, \tag{20}$$

which yields, after multiplication by R_L and some rearrangement, the relation

$$R_L C_s \frac{\mathrm{d}V_\mathrm{out}}{\mathrm{d}t} + V_\mathrm{out} = I_A R_L. \tag{21}$$

If one divides the above equation by $R_L C_s$ and sets $R_L C_s = \tau$, one gets

$$\frac{\mathrm{d}V_\mathrm{out}}{\mathrm{d}t} + \frac{1}{\tau}V_\mathrm{out} = \frac{1}{C_s}I_A, \tag{22}$$

which is Eq. (1) of the examination paper

It will be more convenient to simplify the notation and rephrase Eq. (21) as

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \alpha y = \beta I_A, \tag{23}$$

by setting $\alpha = 1/\tau$ and $\beta = 1/C_s$.

Part b: The shape of the current pulse in Fig. 3 can be described as follows:

$$I_A = \begin{cases} \dfrac{I_0}{T}t & \text{if } 0 \le t \le T, \\ 0 & \text{if } t > T. \end{cases} \tag{24}$$

Let us examine the values of the time constant and the duration of the pulses:

1. With $R_L = 50\ \Omega$, $\tau = 1$ ns, and $T = 10$ ns,
2. With $R_L = 5\ \mathrm{K}\Omega$, $\tau = 100$ ns, and $T = 10$ ns,
3. With $R_L = 5\ \mathrm{K}\Omega$, $\tau = 100$ ns, and $T = 1\ \mu$s.

In cases 1 and 3, the time constant is one-tenth of the pulse width. For these cases, we can find a good approximation Eq. (23) by taking the following steps. We first write this equation as

$$y + \tau\frac{\mathrm{d}y}{\mathrm{d}t} = \tau\beta I_A(t), \tag{25}$$

and then replace the right-hand side by $y(t+\tau)$. This leads to the approximation

$$y(t) = \tau\beta I_A(t-\tau) = \tau\beta I_0 \frac{t-\tau}{T}. \tag{26}$$

Now, the time constant is 10 times smaller than the pulse width in the first and the last case; this means that if we choose t/T as the abscissa and $y/(I_0\tau\beta)$ as the ordinate, it will be enough to draw one figure for the two cases. Apart from an initial transient phase (see case 2), the output will rise linearly with the input. At the end of the pulse, the output will be

$$y(t=T) = \tau\beta I_A(T-\tau) \tag{27}$$

$$= \tau\beta \underbrace{I_0\left(1-\frac{\tau}{T}\right)}_{\text{see Eq. (26)}} = 0.9 I_0\tau\beta. \tag{28}$$

For $t > T$, the output will decay exponentially in accordance with the relation

$$y(t) == 0.9 I_0\tau\beta \exp[-(t-T)/\tau],$$

and would become negligibly small after $t \geq T + 5\tau = 1.5T$. The exact solution is plotted in Fig. 6. In a freehand sketch, one would accept a curved rise and a curved (approximately exponential) fall. If a candidate chooses to argue that $y(t+\tau)$ can be further approximated as $y(t)$ and concludes that output pulse will have nearly the same shape as I_A, as implied by the (dotted) red curve in Fig. 6; such an answer will be acceptable, but will not deserve full credit.

In Case 2, the time constant τ is much longer than than T.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \beta I_A(t) = \frac{\beta I_0}{T} t, \qquad (0 \leq t \leq T). \tag{29}$$

The output will rise parabolically reaching the value $\beta I_0 T$ at $t = \tau$ (see Fig. 7); thereafter it will decay almost linearly because $\mathrm{e}^{-x} = 1 - x$ (approximately) when $x \ll 1$. The dotted red curve in Fig. 7 has no significance and is shown only for indicating the *shape* of the input *current* pulse,

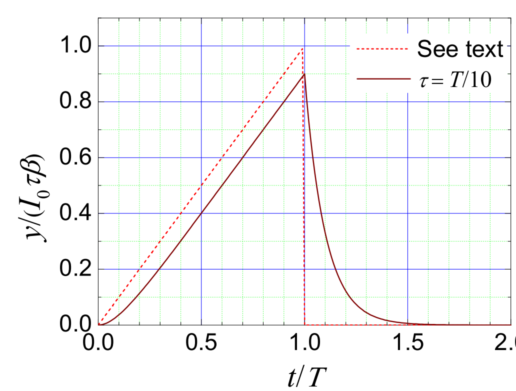

Figure 6: Outputs for cases 1 and 3.

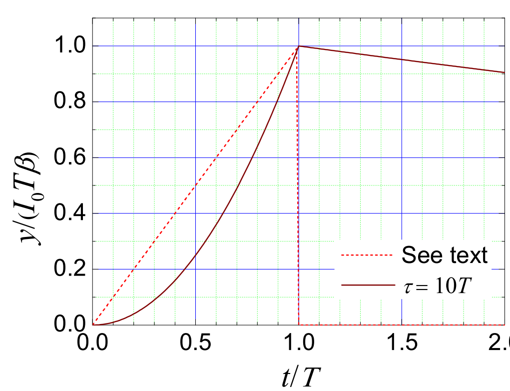

Figure 7: Output for case 2.

Problem 2

1) For marks alloted to different parts, see p. 6. The correct wiring diagram is shown in the figure below. The output of the bridge can be expressed as

$$\frac{V_{\text{out}}}{V_{\text{s}}} = \left[\frac{R_C}{R_C + R_D} - \frac{R_B}{R_A + R_B}\right]. \tag{30}$$

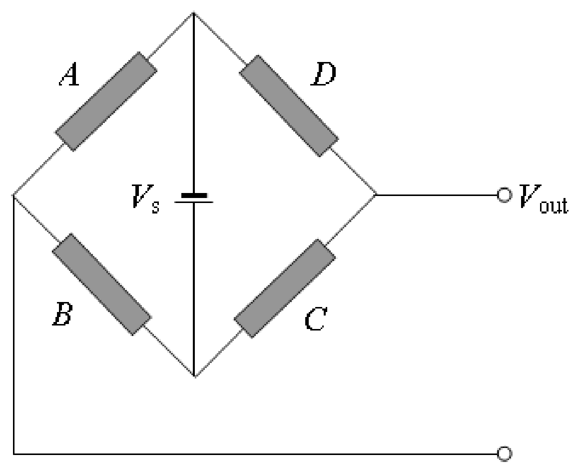

Figure 8: Wiring diagram showing the location of the strain gauges

When all gauges are unstressed, each will have a resistance equal to R, say. If only one gauge, say A, is bonded to the accelerometer, its resistance will change from R to $R+\Delta R = R(1+x)$, where $x = \Delta R/R$ and $|x| \ll 1$. It is easy to see that in this case the output will be

$$\frac{V_{\text{out}}}{V_{\text{s}}} = \left[\frac{1+x}{2(1+\frac{1}{2}x+\cdots)} - \frac{1}{2}\right] \approx \frac{1}{2}[(1+x)(1-\tfrac{1}{2}x)] - \frac{1}{2} = \frac{x}{4}. \tag{31}$$

When all four gauges are used, the resistances of A and C will change from R to $R = R(1+x)$, and those of B and D will change from R to $R = R(1-x)$. If the two end masses move towards the bottom of the page, A and C will be in tension, whereas B and D will be in compression; in other words, $x > 1$. It is easy to see that in this case the output will be

$$\frac{V_{\text{out}}}{V_{\text{s}}} = \left[\frac{1+x}{2} - \frac{1-x}{2}\right] = x. \tag{32}$$

2) A change in temperature will affect all four gauges equally; let y denote the fractional change due to the change in temperature.

$$\frac{V_{\text{out}}}{V_{\text{s}}} = \left[\frac{1+x+y}{2(1+y)} - \frac{1-x+y}{2(1+y)}\right] = \frac{x}{1+y} \approx x. \tag{A4}$$

3) If the acceleration is to the right, gauges C and D will be in compression, and the other two in tension. Each term on the right-hand side will equal $\frac{1}{2}$, and the bridge will remain balanced.

4) Introduce the following symbols:

$$\frac{b}{m} = 2\gamma, \quad \frac{K}{m} = \omega_{\mathrm{n}}^2, \quad \text{so that} \quad \frac{b^2}{mK} = \frac{4\gamma^2}{\omega_{\mathrm{n}}^2}. \tag{33}$$

We consider only one cantilever; for the whole instrument, a factor of two must be applied to describe the total mass, total damping, and total spring constant.

The right-hand side Eq. (1) of the examination paper should read $-m\ddot{x}$, where x denotes the displacement of the frame. Divide by m and rephrase it in terms of the symbols defined in Eq. (33) and the D-operator, obtaining thereby the equation shown below

$$(D^2 + 2\gamma D + \omega_n^2)y = -D^2 x. \tag{34}$$

We now consider a sinusoidal displacement $x = x_0 \sin \omega t = \Im x_0 \mathrm{e}^{\mathrm{i}\omega t}$, so that the acceleration $\alpha \equiv D^2 x = \Im \alpha_0 \mathrm{e}^{\mathrm{i}\omega t}$, with $\alpha_0 = -\omega^2 x_0$. We go on to calculate the particular integral y_{p} as follows

$$y_{\mathrm{p}} = \Im \left[\frac{1}{D^2 + 2\gamma D + \omega_{\mathrm{n}}^2} \{-\alpha_0 \mathrm{e}^{\mathrm{i}\omega t}\} \right] \tag{35}$$

$$= -\alpha_0 \Im \left[\frac{1}{-\omega^2 + 2\mathrm{i}\omega\gamma + \omega_{\mathrm{n}}^2} \mathrm{e}^{\mathrm{i}\omega t} \right] \tag{36}$$

$$= -\alpha_0 \Im \left[\frac{1}{(\omega_{\mathrm{n}}^2 - \omega^2) + 2\mathrm{i}\omega\gamma} \mathrm{e}^{\mathrm{i}\omega t} \right] \tag{37}$$

$$= -\alpha_0 \Im \left[\frac{1}{R\mathrm{e}^{\mathrm{i}\phi}} \mathrm{e}^{\mathrm{i}\omega t} \right] = \frac{-\alpha_0}{R} \sin(\omega t - \phi) = y_0 \sin(\omega t - \phi) \tag{38}$$

where

$$R \equiv \sqrt{(\omega_{\mathrm{n}}^2 - \omega^2)^2 + 4\gamma^2\omega^2}, \quad \text{and } \tan\phi \equiv \frac{2\gamma}{\omega} \tag{39}$$

Whence follows the relation

$$\left| \frac{y_{\mathrm{p}}}{\alpha_0} \right| \equiv S = \frac{1}{\sqrt{(\omega_{\mathrm{n}}^2 - \omega^2)^2 + 4\gamma^2\omega^2}} = \frac{1}{\sqrt{\omega_{\mathrm{n}}^4 + \omega^4 + 2\omega^2(2\gamma^2 - \omega_{\mathrm{n}}^2)}} \tag{40}$$

$$= \frac{1}{\sqrt{\omega_{\mathrm{n}}^4 + \omega^4}}, \quad \text{if } 2\gamma^2 = \omega_{\mathrm{n}}^2 \text{ or } b^2/(mK) = 2. \tag{41}$$

When $2\gamma^2 = \omega_{\mathrm{n}}^2$ we can express the sensitivity S in terms of the ratio $\rho \equiv \omega/\omega_{\mathrm{n}}$ as follows:

$$\omega_{\mathrm{n}}^2 S = \frac{1}{\sqrt{1 + \rho^4}} \tag{42}$$

when $\rho = 0.01$, $\omega_{\mathrm{n}}^2 S$ is practically equal to unity. When $\rho = 1/2$, we can write

$$\omega_{\mathrm{n}}^2 S = \frac{1}{\sqrt{1 + \frac{1}{16}}} \approx \frac{1}{1 + \frac{1}{32}} \approx 1 - \frac{1}{32} = \frac{31}{32}, \quad \text{equal to 1 within 3\%.}$$

We have used Eq. (12) of the examination paper according to which $(1+x)^{-1/2}$ may be replaced by $1 - \frac{1}{2}x$, when $x \ll 1$.

5) The statement is true, since a glance at Eq. (42) shows that the sensitivity is proportional to $1/\omega_{\mathrm{n}}^2$. The disadvantage of having a low natural frequency is that one would need a large mass and a stiff spring; these qualities can only be obtained in a bulky instrument.

Problem 3

Part a: Since we need to measure the true spectral distribution (either E_λ or Q_λ), we must choose a non-selective detector. This means that if a candidate chooses a PMT (or a CCD or any other selective detector) should expect to receive no credit for such an answer. Even with the right choice, some comments should be made about the wavelength range where a particular device would be expected to function properly.

1. A satisfactory answer would mention four non-selective thermal detectors: thermopile, Golay cell, pyroelectric detector, and bolometer. These sensors have been described in Chapter 15 of the lecture notes. All these devices can work over the given spectral range (namely 200–1200 nm)

2. Since the light source has a steady output, one can record several spectra and average these. Alternatively, one can use a chopper and a lock-in amplifier to improve the signal to noise ratio.

3. For measuring Q_λ, one can use a relative quantum counter, which essentially consists of two parts: a highly concentrated solution of a substance that has a high quantum yield of fluorescence and a detector that can detect the fluorescence emitted by the solution. Since most fluorescence substances are transparent to the near infrared region, one would not be able to cover the entire wavelength range of interest. For detecting the emitted radiation, the most convenient device is a PMT (but a Geiger counter or a CCD detector can also be used).

4. Since the energy of a quantum of frequency ν is $h\nu = hc/\lambda$, where h denotes Planck's constant and c is the speed of light, the relation is $E_\lambda = h\nu Q_\lambda = hcQ_\lambda/\lambda$.

5. The relation is $|Q_\lambda \, \mathrm{d}\lambda| = |Q_\sigma \, \mathrm{d}\sigma|$; that is $Q_\sigma = \lambda^2 Q_\lambda$.

Part b: It follows from the given reaction scheme that

$$\phi_X = \frac{k_1}{k_1 + k_2}. \tag{43}$$

Since $\phi_X = 0.05$ and $k_2 = 9.5 \times 10^7 \text{ s}^{-1}$, we find $k_1 = 5 \times 10^6 \text{ s}^{-1}$, so that $k \equiv k_1 + k_2 = 10^8 \text{ s}^{-1}$. A quick way to proceed now is to note that ϕ must equal $\phi_{\mathrm{ET}}\phi_Y$, where

$$\phi_{\mathrm{ET}} = \frac{k_3[Y^0]}{k_1 + k_2 + k_3[Y^0]} \tag{44}$$

is the efficiency of energy transfer. With $\phi = 0.4$ and $\phi_Y = 0.8$, we get $\phi_{\mathrm{ET}} = 0.5$, from which it follows that $k_3[Y^0] = k = 10^8 \text{ s}^{-1}$. Thus $k_3 = (10^8/[Y^0]) \text{ M}^{-1} \text{ s}^{-1}$. To go further, we need $[Y^0]$, which is not given. [A candidate who has omitted $[Y^0]$ in Eq. 44 or an equivalent equation will not get any credit.]