The $\lambda\phi^3$ theory.
Consider the theory of a scalar field ϕ with mass m and a $\frac{\lambda}{3!}\phi^3$ self-interaction in d dimensions.
a.) Write down the Lagrange density $\mathscr{L}$ and explain your choice of signs.
b.) Determine the dimension of the field ϕ and of the coupling λ in d dimensions. Fix d such that the coupling λ is dimensionless.
c.) Draw the Feynman diagram(s) and write down the analytical expression for the self-energy $\mathrm{i}\Sigma$ (i.e. the one-loop correction for the free propgator) in momentum space.
d.) Determine the symmetry factor of $\mathrm{i}\Sigma$.
e.) Calculate the self-energy $\mathrm{i}\Sigma$ using dimensional regularisation.
f.) Determine the running of the mass $m(\mu)$.

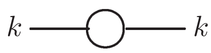
a.) The free Lagrangian is

$$\mathscr{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

the relative sign is fixed by the relativistic energy-momentum relation, the overall sign by the requirement that the Hamiltonian is bounded from below. As the self-interaction is odd, adding $+\frac{\lambda}{3!}\phi^3$ or $-\frac{\lambda}{3!}\phi^3$ is equivalent: both choices will lead to an unstable vacuum. In order to reproduce the Feynman rule, we should choose $\mathscr{L}_I = -\frac{\lambda}{3!}\phi^3$.

b.) The action $S = \int \mathrm{d}^n x \mathscr{L}$ has to be dimensionless. Thus $[\mathscr{L}] = m^n$, $[\phi] = m^{(d-2)/2}$, and thus the coupling is dimensionless in $d = 6$.

c.) Using the Feynman rules gives for



in momentum space

$$\mathrm{i}\Sigma(k^2) = S\,(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^6 p}{(2\pi)^6} \frac{\mathrm{i}}{(p+k)^2 - m^2 + \mathrm{i}\varepsilon} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

where the symmetry factor S is determined in d.) and the vertex $-\mathrm{i}\lambda$ was used.

d.) The self-energy is a second order diagram, corresponding to the term

$$\frac{1}{2!}\left(\frac{-\mathrm{i}\lambda}{3!}\right)^2 \int \mathrm{d}^4 y_1 \mathrm{d}^4 y_2 \langle 0|T[\phi(x_1)\phi(x_2)\phi^3(y_1)\phi^3(y_2) + (y_1 \leftrightarrow y_2)$$

in the perturbative expansion in coordinate space. The exchange graph $y_1 \leftrightarrow y_2$ is identical to the original one, canceling the factor $1/2!$ from the Taylor expansion. We count the number of possible ways to combine the fields in the time-ordered product into four propagators. We have three possibilities to contract $\phi(x_1)$ with a $\phi(y_1)$. Similiarly, there are three possibilities for $\phi(x_2)\phi(y_2)$. The remaining pairs of $\phi(y_1)$ and $\phi(y_2)$ can be contracted in 2! ways. Thus the symmetry factor is

$$S = \left(\frac{1}{2!} \times 2\right)\left(\frac{1}{3!}\right)^2 (3 \times 3 \times 2!) = \frac{1}{2}$$

e.) We combine the two propagators (suppressing the iε) using (9),

$$\frac{1}{(p+k)^2-m^2}\frac{1}{p^2-m^2}=\int_0^1 \mathrm{d}x \frac{1}{D^2}$$

with

$$\begin{aligned} D &= x[(p+k)^2-m^2]+(1-x)(p^2-m^2) \\ &= (p+xk)^2+x(1-x)k^2-m^2=q^2+f\,, \end{aligned}$$

where we introduced $q=p+xk$ as new integration variable and set $f=x(1-x)k^2-m^2$. We go now to $d=2\omega=6-\varepsilon$ dimensions,

$$\mathrm{i}\Sigma(k^2)=\frac{1}{2}\lambda^2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q+f)^2}.$$

Evaluating the integral with (10), using $\Gamma(2)=1$ and $\omega=3-\varepsilon/2$ gives

$$i\Sigma(k^2)=\frac{1}{2}\lambda^2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(q+f)^2}.$$

Evaluating the integral with (10), using $\Gamma(2)=1$ and $\omega=3-\varepsilon/2$ gives

$$\Sigma(k^2)=-\frac{\lambda^2}{2}\frac{\Gamma(-1+\varepsilon/2)}{(4\pi)^3}\int_0^1 \mathrm{d}x f\left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2}.$$

Here, we added a mass scale μ in order to make the ε dependent term dimensionless such that we can expand it using (11),

$$\left(\frac{4\pi\mu^2}{f}\right)^{\varepsilon/2}=1+\frac{\varepsilon}{2}\ln\left(\frac{4\pi\mu^2}{f}\right)+\mathcal{O}(\varepsilon^2).$$

Expanding also

$$\Gamma(-1+\varepsilon/2)=-\left[\frac{2}{\varepsilon}+1-\gamma+\mathcal{O}(\varepsilon)\right]$$

we arrive at

$$\Sigma(k^2)=\frac{\alpha}{2}\left[\left(\frac{2}{\varepsilon}+1-\gamma\right)\left(\frac{k^2}{6}-m^2\right)+\int_0^1 \mathrm{d}x f\ln\left(\frac{4\pi\mu^2}{f}\right)\right]$$

where we used $\int_0^1 \mathrm{d}x f=k^2/6-m^2$ and set $\alpha=\lambda^2/(4\pi)^3$. The obtained expression for the self-energy has the UV divergence isolated into an $1/\varepsilon$ pole which is ready for subtraction.

f.) The self-energy connects the bare and the physical mass. Subtracting the poles leads to an ambiguity, since we can subtract also an arbitrary finite constant. (This corresponds to the choice of a "renormalisation scheme.") We choose to subtract the pole plus $\gamma-\ln(4\pi)$ ("$\overline{\mathrm{MS}}$ scheme")

$$m_{\text{phys}}^2=m^2-\frac{1}{2}\alpha\left[\frac{m^2}{6}-m^2+\int_0^1 \mathrm{d}x f_0\ln\left(\frac{\mu^2}{f_0}\right)\right]$$

with $f_0 = [1 - x(1-x)]m^2$. Performing the x integral gives

$$m_{\rm phys}^2 = m^2 \left[1 + \frac{5}{12}\alpha \left(\ln(\mu^2/m^2) + \text{const.} \right) \right]$$

Physics has to be independent of μ, and thus the explicit μ dependence has to be cancelled by an implicit μ dependence of m (converting it into a "running parameter".) Taking a loagrithmic derivative w.r.t. μ of $\ln(m_{\rm phys})$ and using that α runs only at $\mathcal{O}(\alpha^2)$ gives

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln(\mu)} \ln(m_{\rm phys}) = \frac{1}{m}\frac{\mathrm{d}m}{\mathrm{d}\ln(\mu)} + \frac{5}{12}\alpha + \mathcal{O}(\alpha^2)$$

or

$$\frac{\mathrm{d}m}{\mathrm{d}\ln(\mu)} = -\frac{5}{12}\alpha m^2 + \mathcal{O}(\alpha^2).$$

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The $\lambda\phi^3$ theory is discussed at length in Srednicki, where you can find more details.