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Solution to the exam in FY3464 QUANTUM FIELD THEORY I

Wednesday october 17, 2007

This solution consists of 4 pages.

Problem 1.

Consider the model defined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \lambda\varphi\boldsymbol{E}\cdot\boldsymbol{B}, \tag{1}$$

where φ is a real scalar field, $\boldsymbol{E} = -\dot{\boldsymbol{A}}$, and $\boldsymbol{B} = \boldsymbol{\nabla}\times A$.

a) Find the canonically conjugate field Π_φ of φ.

$$\Pi_\varphi = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = \dot{\varphi}. \tag{2}$$

b) Find the canonically conjugate field $\Pi_{\boldsymbol{A}}$ of $\boldsymbol{A}$.

$$\Pi_{\boldsymbol{A}} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{A}}} = -\frac{\partial\mathcal{L}}{\partial\boldsymbol{E}} = -\lambda\varphi\boldsymbol{B}. \tag{3}$$

c) Find the Hamiltonian density $\mathcal{H}$ of this model.

$$\mathcal{H} = \Pi_\varphi\dot{\varphi} + \Pi_{\boldsymbol{A}}\cdot\boldsymbol{A} - \mathcal{L} = \frac{1}{2}\Pi_\varphi^2 + \frac{1}{2}\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\varphi + \frac{1}{2}m^2\varphi^2. \tag{4}$$

d) We use natural units. What is the mass dimension of the coupling parameter λ:
(i) In 4 space-time dimensions? (ii) In d space-time dimensions?

By comparing dimensions between the first two terms in the Lagrangian (1),

$$\left[m^2\,\varphi^2\right] = [m]^2\,[\varphi]^2 = [\partial_\mu\varphi\,\partial^\mu\varphi] = \ell^{-2}\,[\varphi]^2\,,$$

we note that mass and length dimensions are inverse in natural units (as is also obvious from the expression for Compton wavelength). From the fact that the action $S = \int \mathrm{d}^d x\,\mathcal{L}$ must be dimensionless (dimension of $\hbar$) it follows that $\mathcal{L}$ must have dimension $\ell^{-d} = [\partial_\mu\varphi\,\partial^\mu\varphi] = [\varphi]^2\,\ell^{-2}$, i.e. that

$$[\varphi] = \ell^{(2-d)/2} = [m]^{(d-2)/2}\,.$$

The dimension of $\boldsymbol{E}\cdot\boldsymbol{B}$ cannot be read out of the Lagrangian (1), although it can be seen that $[\boldsymbol{E}] = [\boldsymbol{B}]$ from their relations to $\boldsymbol{A}$. This was an oversight in the exam set; it was assumed that the dimensions are the same as in the "standard" case, when there is a term $\mathcal{L}_{\text{Maxwell}} = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right)$ in the Lagrangian. This leads to the result that that $[\boldsymbol{E}] = [\boldsymbol{B}] = m^{d/2}$. It now follows that

$$[\lambda] = [m]^d\,[\varphi]^{-1}\,[\boldsymbol{E}\cdot\boldsymbol{B}]^{-1} = [\varphi]^{-1} = [m]^{(2-d)/2}\,. \tag{5}$$

I.e, the mass dimension is -1 in the case when $d = 4$.

e) Find the Euler Lagrange equation for φ.

We find that

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \partial^\mu\varphi \quad\text{and}\quad \frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi + \lambda\boldsymbol{E}\cdot\boldsymbol{B},$$

so that the Euler Lagrange equation,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi},$$

becomes

$$\left(\Box + m^2\right)\varphi = \lambda\boldsymbol{E}\cdot\boldsymbol{B}. \tag{6}$$

f) Find the Euler Lagrange equation for $\boldsymbol{A}$.

It is convenient to first write

$$\boldsymbol{E}\cdot\boldsymbol{B} = -\varepsilon^{\ell jk}\dot{A}^\ell\,\partial_j A^k = -\varepsilon^{jk\ell}\dot{A}^j\,\partial_k A^\ell,$$

so that we find

$$\frac{\partial\mathcal{L}}{\partial\dot{A}^\ell} = -\lambda\varphi\,\varepsilon^{\ell jk}\partial_j A^k = -\lambda\varphi\,B^\ell \quad\text{and}\quad \frac{\partial\mathcal{L}}{\partial_k A^\ell} = -\lambda\varphi\,\varepsilon^{jk\ell}\dot{A}^j.$$

Thus, the Euler-Lagrange equation becomes

$$-\lambda\left[\partial_0(\varphi\,B^\ell) - \varepsilon^{\ell kj}\partial_k(\varphi\dot{A}^j)\right] = -\lambda\left(\dot{\varphi}B^\ell + \varepsilon^{\ell kj}E^j\partial_k\varphi\right) - \lambda\varphi\left[\dot{B}^\ell - (\boldsymbol{\nabla}\times\dot{\boldsymbol{A}})^\ell\right].$$

The last term on the right vanishes. Assuming $\lambda \neq 0$ we arrive at the equation

$$\boldsymbol{B}\dot{\varphi} - \boldsymbol{E}\times\boldsymbol{\nabla}\varphi = 0. \tag{7}$$

Comment: By introducing the electromagnetic *field tensor* $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, and its *dual field tensor* $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$, this equation can be written in manifestly covariant form[1],

$$\tilde{F}^{\mu\nu}\,\partial_\nu\varphi = 0. \tag{8}$$

[1] We have the relations $E^i = F^{0i} = -\frac{1}{2}\varepsilon^{ijk}\,\tilde{F}^{jk}$, and $B^i = \tilde{F}^{0i} = \frac{1}{2}\varepsilon^{ijk}\,F^{jk}$. I.e. the duality transformation $F^{\mu\nu} \to \tilde{F}^{\mu\nu}$ amounts to $(\boldsymbol{E},\boldsymbol{B}) \to (\boldsymbol{B},-\boldsymbol{E})$.

g) The Lagrangian density $\mathcal{L}$ is invariant under the transformation

$$\boldsymbol{A}(\boldsymbol{x},t) \to \boldsymbol{A}'(\boldsymbol{x},t) = \boldsymbol{A}(\boldsymbol{x},t) + \boldsymbol{\nabla}\Lambda(\boldsymbol{x}),$$

for all differentiable functions $\Lambda(\boldsymbol{x})$. Use the Nöther theorem to find the corresponding conserved Nöther current J_Λ.

The general expression for the Nöther current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_a}\,\delta\Phi_a, \tag{9}$$

where Φ_a runs over all available fields. Here we have $\delta\varphi = 0$ and $\delta\boldsymbol{A} = \boldsymbol{\nabla}\Lambda$. Thus we find

$$J^0_\Lambda = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{A}}} \cdot \boldsymbol{\nabla}\Lambda = -\lambda\varphi\,\boldsymbol{B}\cdot\boldsymbol{\nabla}\Lambda, \tag{10}$$

$$J^k_\Lambda = \lambda\varphi\,\varepsilon^{jk\ell}E^j\partial_\ell\Lambda = -\lambda\varphi\,(\boldsymbol{E}\times\boldsymbol{\nabla}\Lambda)^k\,. \tag{11}$$

Comment: By introducing the dual field tensor this can also be written in manifestly covariant form

$$J^\mu_\Lambda = -\lambda\varphi\,\tilde{F}^{\mu\nu}\partial_\nu\Lambda. \tag{12}$$

Problem 2.

The field expansion of the free electromagnetic field in Coulomb gauge is

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2|\boldsymbol{k}|V}} \left(a_{\boldsymbol{k},r}\, \hat{e}_{\boldsymbol{k},r}\, \mathrm{e}^{-ikx} + \text{hermitian conjugate} \right). \tag{13}$$

Then the matrix element $\langle \Omega |\, a_{\boldsymbol{q},s}\, \boldsymbol{A}(x)\, | \Omega \rangle$ equals

A. 0 ☐

B. $\frac{1}{\sqrt{2|\boldsymbol{q}|V}}\, \hat{e}_{\boldsymbol{q},s}\, \mathrm{e}^{-iqx}$ ☐

C. $a_{\boldsymbol{q},s}$ ☐

D. $\frac{1}{\sqrt{2|\boldsymbol{q}|V}}\, \hat{e}^{*}_{\boldsymbol{q},s}\, \mathrm{e}^{iqx}$ X

E. None of the alternatives above. ☐

Problem 3.

Let $\mathcal{T}$ be the time ordering operator, and $\varphi(x)$, $\varphi^\dagger(x)$ quantized complex Klein Gordon fields. Then we have (in natural units, i.e. when $\hbar = c = 1$)

A. $\mathcal{T}\left\{\varphi(x)\varphi^\dagger(y)\right\} = \mathcal{T}\left\{\varphi^\dagger(y)\varphi(x)\right\}$ X

B. $\mathcal{T}\left\{\varphi(x)\varphi^\dagger(y)\right\} = \mathcal{T}\left\{\varphi^\dagger(y)\varphi(x)\right\} + \mathrm{i}G_F(x-y)$ ☐

C. $\mathcal{T}\left\{\varphi(x)\varphi^\dagger(y)\right\} = \mathcal{T}\left\{\varphi^\dagger(y)\varphi(x)\right\} - \mathrm{i}G_F(x-y)$ ☐

D. $\mathcal{T}\left\{\varphi(x)\varphi^\dagger(y)\right\} = -\mathcal{T}\left\{\varphi^\dagger(y)\varphi(x)\right\} - \mathrm{i}G_F(x-y)$ ☐

E. None of the alternatives above. ☐

Here $G_F(x-y)$ is the Feynman propagator for a complex Klein Gordon field.

Problem 4.

The Dirac equation

$$\left[\mathrm{i}\left(\gamma^0\partial_0 + \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\right) - m\right]\psi(x^0,\boldsymbol{x}) = 0$$

is invariant under space inversion (parity transformation), $\boldsymbol{x} \to -\boldsymbol{x}$. I.e, if $\psi(x^0,\boldsymbol{x})$ solves the Dirac equation then so does $\psi_P(x^0,\boldsymbol{x})$, where

A. $\psi_P(x^0,\boldsymbol{x}) = \mathrm{i}\gamma^2\, \psi^*(x^0,-\boldsymbol{x})$ ☐

B. $\psi_P(x^0,\boldsymbol{x}) = \gamma^1\,\gamma^3\, \psi^*(x^0,-\boldsymbol{x})$ ☐

C. $\psi_P(x^0,\boldsymbol{x}) = \psi(x^0,-\boldsymbol{x})$ ☐

D. $\psi_P(x^0,\boldsymbol{x}) = \gamma^0\, \psi(x^0,-\boldsymbol{x})$ X

E. $\psi_P(x^0,\boldsymbol{x}) = \psi^*(-x^0,-\boldsymbol{x})$ ☐