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Contact during exam:
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**Exam in FY8304 Mathematical Approximation Methods in Physics**
Thursday 9. December 2004
Time: 09:00—14:00

Exam aids: (Alternative B): Approved calculater.

This exam consists of 2 pages and a general addendum of 1 page.

## Problem 1:

Consider the differential equation

$$x^4\, y''(x) + x^3\, y'(x) - y(x) \;=\; 0.$$

**a)** Find and classify the singular points for this equation.

**b)** Determine the possible leading asymptotic behavior for $y(x)$ when

i. $x \to 0$.
ii. $x \to 1$.
iii. $x \to \infty$.

## Problem 2:

Determine leading behavior of the integral

$$I(a) = \int_0^\infty dt\, \mathrm{e}^{-t+at^{-1}}\, t^a,$$

for

**a)** the limit $a \to 0$,

**b)** the limit $a \to \infty$.

**Problem 3:**
Consider the boundary value problem

$$\frac{1}{100}\,y''(x) + (x^2+1)y'(x) - x^3 y(x) \;=\; 0, \qquad y(0) = y(1) = 1. \tag{1}$$

Introduce the perturbation parameter $\epsilon = 1/100$ in the first term, and use the boundary layer method to find

**a)** The outer and inner regions, and how the thickness of any boundary layer scales with $\epsilon$.

**b)** The outer and inner equations, and their solutions.

**c)** The condition for matching the outer and inner solution, and the uniform approximation.

**d)** Make a sketch of the uniform approximation to Equation (1).

**Problem 4:**
The WKB solution to the eigenvalue problem

$$\epsilon^2\, y''(x) + Q(x,E)y(x) \;=\; 0 \;,$$

where $Q(x) = V(x) - E$, has the leading order solution

$$y(x) \approx \frac{1}{[Q(x,E)]^{1/4}}\; \exp\Big(\pm\frac{1}{\epsilon}\int^x dt\,\sqrt{Q(t)}\Big)\,.$$

**a)** Show that the connection formula from region I (with $Q(x) > 0$ to region III (with $Q(x) < 0$) is

$$\begin{aligned} y_{III} \;&=\; 2C\,\frac{1}{[-Q(x,E)]^{1/4}}\; \sin\Big(\frac{1}{\epsilon}\int_x^{x_0} dt\,\sqrt{-Q(t)} + \frac{\pi}{4}\Big) \\ \leftarrow\; y_I \;&=\; C\,\frac{1}{[Q(x,E)]^{1/4}}\; \exp\Big(\frac{-1}{\epsilon}\int_{x_0}^{x} dt\,\sqrt{Q(t)}\Big). \end{aligned}$$

Assume that the two regions are connected by a region II, where $Q(x)$ is approximately linear.

**b)** Use the connection formula to find the WKB eigenvalue condition.

**c)** Consider bound states ($E < 0$) in the potential $V(x)$ given by

$$V(x) \;=\; -\frac{V_0}{\cosh^2 x}\,,$$

and find the WKB approximation to the eigen-values $E_n$. Are there any limitations of the quantum number $n$?

# Some information that may be useful

## Airy functions

$$
\begin{aligned}
\mathrm{Ai}(x) &\approx \frac{1}{2\sqrt{\pi}}\, x^{-1/4} \exp(-\frac{2}{3}x^{3/2}) & x \to \infty \\
\mathrm{Bi}(x) &\approx \frac{1}{\sqrt{\pi}}\, x^{-1/4} \exp(\frac{2}{3}x^{3/2}) & x \to \infty \\
\mathrm{Ai}(x) &\approx \frac{1}{\sqrt{\pi}}\, (-x)^{-1/4} \sin(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}) & x \to -\infty \\
\mathrm{Bi}(x) &\approx \frac{1}{\sqrt{\pi}}\, (-x)^{-1/4} \cos(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}) & x \to -\infty
\end{aligned}
$$

## A numerical value

$$\sqrt{\frac{2}{e}} = 0.85776...$$

## A WKB integral

$$\int_{-u_0}^{u_0} \sqrt{u_0^2 - u^2}\, \frac{du}{1+u^2} = \pi[\sqrt{u_0^2+1} - 1]$$