NTNU **Institutt for fysikk**

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Exam in FY8304 MATHEMATICAL APPROXIMATION METHODS IN PHYSICS

Wednesday december 13, 2006
09:00–13:00

Allowed help: Alternativ **C**
Standard calculator
K. Rottman: *Matematisk formelsamling* (all languages).
Schaum's Outline Series: *Mathematical Handbook of Formulas and Tables.*

Det finnes også en norsk utgave av dette oppgavesettet. The exam results will be made available on the webpage of the course, `http://web.phys.ntnu.no/~kolausen/FY8304/`, as soon as they are ready.

This problem set consists of 2 pages.

Problem 1. Classification of a differential equation

Consider the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2}{x+1}\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{(x+1)^4}\right] y(x) = 0. \tag{1}$$

a) Find and classify the singular points of this equation.

b) Find the possible leading asymptotic behaviours of $y(x)$ as i) $x \to -1$, ii) $x \to \infty$.

c) Find the general solution of this equation.

Problem 2. Asymptotic expansion of an integral

The integral

$$I(\varepsilon) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2}\, \mathrm{e}^{-\varepsilon x^4} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{1}{2}x^2}\, \mathrm{e}^{-\varepsilon x^4} \tag{2}$$

can be expanded in an asymptotic power series,

$$I(\varepsilon) \sim 1 + \sum_{n=1}^{\infty} a_n\, (-\varepsilon)^n \quad \text{as } \varepsilon \to 0^+. \tag{3}$$

a) Find an integral expression for the coefficients a_n (you don't need to evaluate the integral).

b) Find the asymptotic behaviour of the coefficients a_n as $n \to \infty$.

c) Assume that $\varepsilon = 10^{-10}$. How many terms is it worth including in the asymptotic series for $I(\varepsilon)$?

Given:

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\frac{1}{2}\alpha x^2} = \sqrt{\frac{2\pi}{\alpha}}. \tag{4}$$

Problem 3. A Boundary Layer problem
Consider the boundary value problem

$$\varepsilon y''(x) + (1+x)y'(x) + ay(x) = 0, \quad y(0) = y(1) = 1, \tag{5}$$

in the limit $\varepsilon \to 0^+$.

a) Find the outer solution to the boundary value problem.

b) Determine the position of the boundary layer, and how its thickness scale with ε.

c) Find the inner solution to the boundary value problem.

d) Write down the uniform solution to the boundary value problem.

e) Assume that the boundary value problem instead is

$$\varepsilon y''(x) + xy'(x) + ay(x) = 0, \quad y(0) = y(1) = 1. \tag{6}$$

Find the inner equation, i.e. the equation which describes the boundary layer, for this problem.

Problem 4. A Nonlinear Schrödinger Equation
In this problem you shall analyze a special class of solutions to a nonlinear Schödinger equation,

$$\varepsilon^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(x) + Q(x)\Psi(x) + \lambda |\Psi(x)|^n \, \Psi(x) = 0, \tag{7}$$

for small positive ε.

a) First assume that $\lambda = 0$, $Q(x) > 0$, and find the leading order WKB-solution to the problem.

b) Next assume that $\lambda > 0$, $n > 0$, and that the solution $\Psi(x)$ is such that $|\Psi(x)|$ varies slowly with x also when $\varepsilon \to 0^+$. Find in this case a one-parameter family of leading orden solutions (in implicit form, i.e. the solution may involve expressions defined by an algebraic equation which cannot necessarily be solved explicitly).