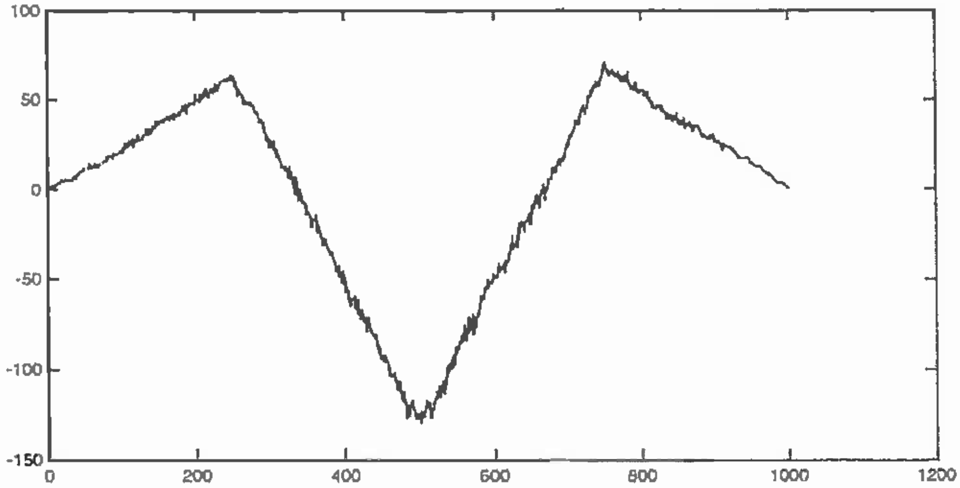
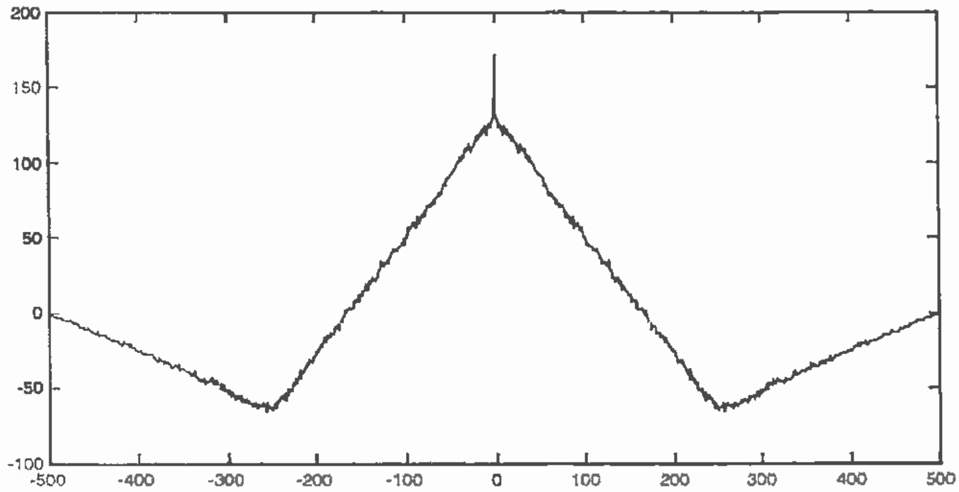
Suggested solutions Examination May 2010



Solution A1



Solution A2

A3. $H(s) = \frac{2}{(s+2)^2}$

Unit step excitation $X(s) = \frac{1}{s}$

$\Rightarrow$ $Y(s) = \frac{2}{s(s+2)^2}$

Partial fraction expansion:

$$\frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

$A = \frac{2}{(0+2)^2} = \frac{1}{2}$; $B = \frac{2}{(-2)} = -1$

Thus, $\frac{0.5}{s} - \frac{1}{(s+2)^2} + \frac{C}{(s+2)} = \frac{2}{s(s+2)^2}$

$$0.5(s+2)^2 - s + Cs(s+2) = 2$$

$$0.5(s^2+4s+4) - s + Cs^2 + 2Cs = 2$$

s^2: $0.5 + C = 0$ $\Rightarrow$ $C = -0.5$

s^1: $2s - s + 2C \cdot s = 0$ ok

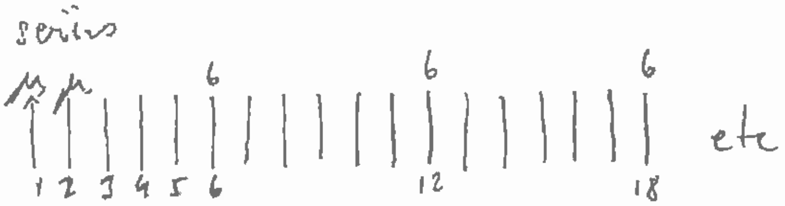
s^0: $2 = 2$ ok

$\therefore$ $Y(s) = \frac{0.5}{s} - \frac{1}{(s+2)^2} - \frac{0.5}{(s+2)}$

from table:

$$y(t) = \left(0.5 - 0.5 \cdot e^{-2t} - t e^{-2t}\right) \cdot \epsilon(t)$$

A4.



for a series, the expected random number μ appears @ throw $1 \to 5$

$$\mu = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

$\Rightarrow$ for series $1 \to 6$ $\quad \frac{6 + 5 \cdot 3.5}{6} = \frac{47}{12} = \overline{\mu_y[k]}\ (\sim 3.92)$

and similar for all series in blocks of 6

Similarly we get

$$E\{y^2[k]\}_{normal} = \frac{1+4+9+16+25+36}{6} = \frac{91}{6}$$

so for the loaded die we get

$$\frac{1}{6}\left(5 \cdot \frac{91}{6} + 6^2\right) = \frac{671}{36} \quad (\sim 18.64)$$

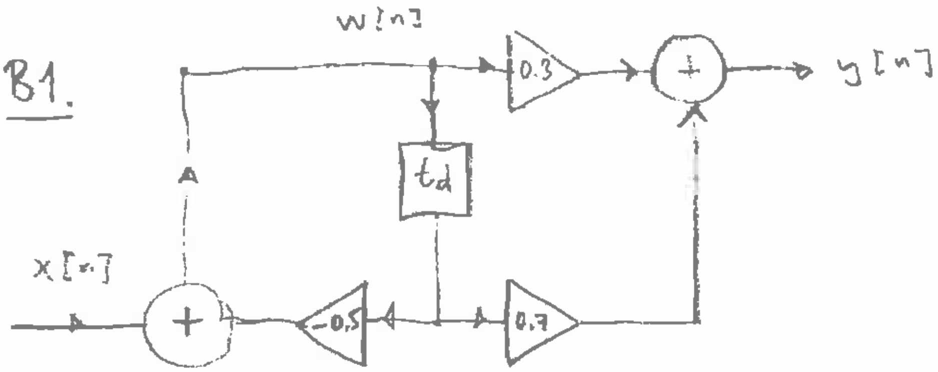
So that for loaded die $\sigma_x^2 = \frac{671}{36} - \left(\frac{47}{12}\right)^2 = \frac{475}{144}$ (~ 3.30)

compare with 'normal' die $\sigma^2 = \frac{91}{6} - 3.5^2 = \frac{35}{12}$ (2.91)

<u>Ans.</u> $\overline{\mu_y[k]} = \frac{47}{12}$ ~ 3.92 ; $\sigma_y[k] = \frac{5\sqrt{19}}{12}$ ~ 1.81

1.708 ; 2.91

B1.



a)

$w[n] = x[n] - 0.5\, w[n-1]$ $\overset{Z}{\curvearrowright}$ $W = X - 0.5 z^{-1} W$

$y[n] = 0.3\, w[n] + 0.7\, w[n-1]$ $\qquad Y = 0.3 W + 0.7 z^{-1} W$

$\Rightarrow \quad W(1 + 0.5 z^{-1}) = X \quad \Rightarrow \quad W = X \cdot \frac{1}{(1 + 0.5 z^{-1})}$

$Y = (0.3 + 0.7 z^{-1})\, W$

$\Rightarrow \quad Y = \frac{(0.3 + 0.7 z^{-1})}{(1 + 0.5 z^{-1})}\, X$

$Y\,(1 + 0.5 z^{-1}) = 0.3 X + 0.7 z^{-1} X$ $\quad \overset{z^{-1}}{\curvearrowleft}$

$\Rightarrow \quad \underline{y[n] = 0.3\, x[n] + 0.7\, x[n-1] - 0.5\, y[n-1]}$

b) $H(z) = \frac{Y(z)}{X(z)} = \frac{0.3 + 0.7 z^{-1}}{1 + 0.5 z^{-1}} = \frac{0.3 z + 0.7}{z + 0.5}$

B1. cont

c) unit step response

$$Y(z) = H(z)\cdot\frac{z}{(z-1)} = \frac{(0.3z+0.7)z}{(z+0.5)(z-1)}$$

long division gives

$$0.3 + 0.85z^{-1} + 0.575z^{-2} + 0.7125z^{-3} + 0.64375z^{-4}$$

$$z^2 - 0.5z - 0.5 \;\big|\; 0.3z^2 + 0.7z$$

$$0.3z^2 - 0.15z - 0.15$$

$$0 \qquad 0.85z + 0.15$$

$$0.85z - 0.425 - 0.425z^{-1}$$

$$0 \qquad 0.575 + 0.425z^{-1}$$

$$0.575 - 0.2875z^{-1} - 0.2875z^{-2}$$

$$0 \qquad 0.7125z^{-1} + 0.2875z^{-2}$$

$$0.7125z^{-1} - 0.35625z^{-2}$$

$$0 \qquad -0.35625z^{-3}$$

$$0.64375z^{-2} + 0.35625z^{-3}$$

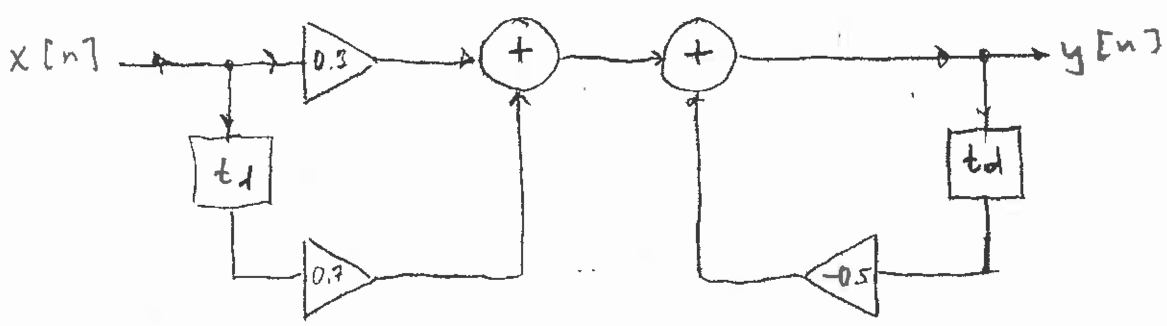
$$0.64375z^{-2} - 0.321875z^{-3} - 0.321875z^{-4}$$

$$0 \qquad 0.678125z^{-3} + 0.321875z^{-4}$$

$\Rightarrow$ Sequence: $\{\underset{\uparrow}{0.3}, 0.85, 0.575, 0.7125, \ldots\}$

d) the difference equation in (a) gives

$$y[n] = 0.3\,x[n] + 0.7\,x[n-1] - 0.5\,y[n-1]$$



B2

a) Prove the time-shift property of the Fourier transform

$$f(t-t_0) \rightarrow F(\omega)\, e^{-i\omega t_0}$$

From the definition $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$

now let $t \rightarrow t - t_0$

$$F(\omega) = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-i\omega t}\, dt$$

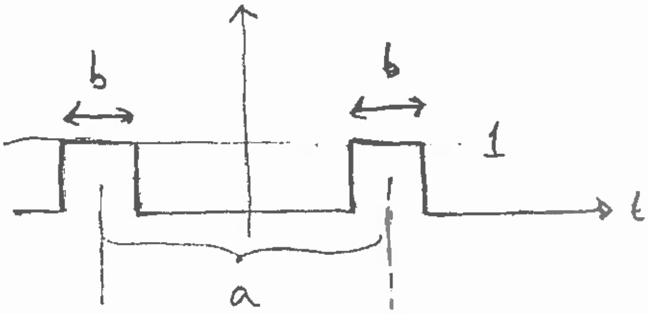
let $t - t_0 = t'$ $\Rightarrow$ $dt' = dt$; limits $\rightarrow \pm\infty$

$$\Rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t')\, e^{-i\omega(t'+t_0)}\, dt' = e^{-i\omega t_0} \underbrace{\int_{-\infty}^{\infty} f(t')\, e^{-i\omega t'}\, dt'}_{F(\omega)}$$

$\therefore$ $e^{-i\omega t_0} \cdot F(\omega)$ Q.E.D

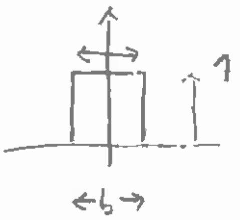
B2

b)



$= f(t)$

The FT is linear, thus we can add the contribution for each time-slot and make use of the shift-theorem as deduced in (a)



$$\mathrm{rect}(t/b) \xrightarrow{FT} b \cdot \mathrm{sinc}\left(\frac{\omega b}{2}\right)$$

shift-theorem gives that

$$F\{f(t)\} = b \cdot \mathrm{sinc}\left(\frac{\omega b}{2}\right) \cdot \left[e^{i\frac{\omega a}{2}} + e^{-i\frac{\omega a}{2}}\right]$$

$$= \underline{2b\cos\left(\frac{\omega a}{2}\right)\mathrm{sinc}\left(\frac{\omega b}{2}\right)}$$

B2 cont...

check by brute force ....

$$\int_{-\frac{a}{2}-\frac{b}{2}}^{-\frac{a}{2}+\frac{b}{2}} e^{-i\omega t}\,dt + \int_{\frac{a}{2}-\frac{b}{2}}^{\frac{a}{2}+\frac{b}{2}} e^{-i\omega t}\,dt = F(\omega) =$$

$$= \frac{1}{(-i\omega)}\Big[e^{-i\omega t}\Big]_{-\frac{a}{2}-\frac{b}{2}}^{-\frac{a}{2}+\frac{b}{2}} + \frac{1}{(-i\omega)}\Big[e^{-i\omega t}\Big]_{\frac{a}{2}-\frac{b}{2}}^{\frac{a}{2}+\frac{b}{2}} =$$

$$= \frac{(-2)}{2i\omega}\left[e^{\frac{i\omega a}{2}}\cdot e^{-\frac{i\omega b}{2}} - e^{\frac{i\omega a}{2}}\cdot e^{\frac{i\omega b}{2}} + e^{-\frac{i\omega a}{2}} e^{-\frac{i\omega b}{2}} - e^{-\frac{i\omega a}{2}} e^{\frac{i\omega b}{2}}\right]$$

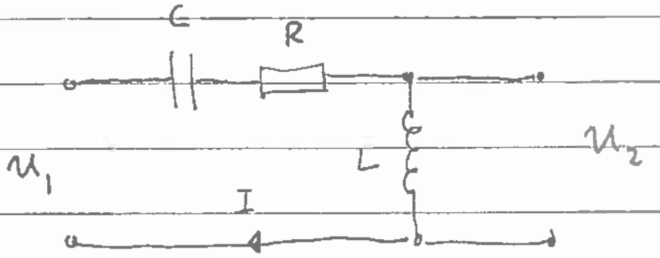
$$= \frac{-2}{2i\omega}\left[e^{\frac{i\omega a}{2}}\left(e^{-\frac{i\omega b}{2}} - e^{\frac{i\omega b}{2}}\right) + e^{-\frac{i\omega a}{2}}\left(e^{-\frac{i\omega b}{2}} - e^{\frac{i\omega b}{2}}\right)\right]$$

$$= \left(\frac{e^{\frac{i\omega b}{2}} - e^{-\frac{i\omega b}{2}}}{2i}\right)\cdot\frac{2}{\omega}\,\underbrace{\frac{2}{2}\left(e^{\frac{i\omega a}{2}} + e^{-\frac{i\omega a}{2}}\right)}_{}$$

$$= \frac{2\cdot 2\frac{b}{2}}{\omega\frac{b}{2}}\cos\left(\frac{\omega a}{2}\right)\cdot\sin\left(\frac{\omega b}{2}\right) = \underline{2b\cdot\cos\left(\frac{\omega a}{2}\right)\,\mathrm{sinc}\left(\frac{\omega b}{2}\right)}$$

Q.E.D

B9:



a) impedance method gives $I = \dfrac{u_1}{\frac{1}{sC} + R + sL}$

$u_2 = I \cdot sL$

$\Rightarrow u_2 = u_1 \cdot \dfrac{sL}{\frac{1}{sC} + R + sL}$

$\Rightarrow \dfrac{u_2}{u_1} = H(s) = \dfrac{s^2 LC}{s^2 LC + sRC + 1} = \dfrac{s^2}{\left(s^2 + s\frac{R}{L} + \frac{1}{LC}\right)}$

b) $H(s) = \dfrac{s^2}{(s+a)^2}$?

$s^2 + s\dfrac{R}{L} + \dfrac{1}{LC} = \left(s + \dfrac{R}{2L}\right)^2 + \dfrac{1}{LC} - \dfrac{R^2}{4L^2}$

holds if $\dfrac{1}{LC} = \dfrac{R^2}{4L^2} \Rightarrow C = \dfrac{4L}{R^2}$

$a = \dfrac{R}{2L}$

B3:

c) $H(\omega) = \dfrac{(i\omega)^2}{(i\omega + a)^2}$

$|H(\omega)| = \dfrac{\omega^2}{(\omega^2 + a^2)} = \cancel{\dfrac{1}{\sqrt{2}}}$ ~~@ cut off~~

$\Updownarrow$

cut-off frequency for $\omega = 100$ $\left(20 \cdot \log H(100) = -3 \right)$

$\Rightarrow \cancel{a = 100} \Rightarrow \cancel{\dfrac{R}{2L} = 100} \Rightarrow \dfrac{R}{L} = \cancel{200}\ 12{,}86$

$a^2 = (\sqrt{2} - 1)\, 100 \Rightarrow 6.43 = \dfrac{R}{2L}$

d) plot $|H(\omega)|$ log scale $0.1; 1; 10; 100; 1000$

0.1	$20 \cdot \log \dfrac{0.1}{0.1 + 100} \sim 20 \cdot (-3) = -60$
1.0	$20 \cdot \log \dfrac{1}{1 + 100} \sim 20\ (-2) = -40$
10	$20 \cdot \log \dfrac{10}{10 + 100} \sim 20\ (-1) = -20$
100	$20 \cdot \log \dfrac{100}{100 + 100} \sim \quad = -3$
1000	$20 \cdot \log \dfrac{1000}{1000 + 100} \sim 20 \cdot 0 = 0$