Norwegian University of Science and Technology
Department of Physics

Contacts during the exam:
Pawel Sikorski, phone: 98486426

EXAM
TFY4280 Signal Processing

Sat. 2 June 2012. 09:00

Examination support materials:

- Simple calculator (according to NTNU exam regulations)
- K. Rottmann: Matematisk formelsamling (eller tilsvarende)
- Carl Angell og Bjørn Ebbe Lian: Fysiske størrelser og enheter, navn og symboler (eller tilsvarende)

Answer must be written in English or Norwegian. Number of points given to each sub-question is given in bold font. The maximum score for the exam is **100p**. The exam consists of 4 questions. Attachment: 2 pages with transform tables and properties.

Q1 (25p)

A) (15p) Calculate response (output $y(t)$) for a unit step function input ($\varepsilon(t)$) and delta impulse input $\delta(t)$ from a given impulse response function h_1 in the time domain:

$$h_1(t) = \left(e^{-t} - e^{-2t}\right)\varepsilon(t)$$

B) (10p) How would you describe the output of this LTI system, when a random signal $x(t)$ described by its μ_x and $\varphi_{xx}(\tau)$ is the input signal. Explain briefly and calculate μ_y.

A) For $x(t) = \delta(t)$

$$y(t) = h_1(t)$$

For $x(t) = \varepsilon(t)$ we need to find Laplace transform of $h_1(t)$:

$$\mathscr{L}\{h_1(t)\} = \mathscr{L}\{\left(e^{-t} - e^{-2t}\right)\varepsilon(t)\} = \frac{1}{1+s} - \frac{1}{s+2} = \frac{s+2-s-1}{(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$

$$X(s) = \frac{1}{s}$$

$$Y(s) = H(s)X(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{s+2}$$

$$A = \frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}$$

check:

$$\frac{0.5}{s} - \frac{1}{(s+1)} + \frac{0.5}{(s+2)} = \frac{0.5(s+2)(s+1) - s(s+2) + 0.5s(s+1)}{s(s+1)(s+2)} =$$

$$\frac{0.5s^2 + 3/2s + 1 - s^2 - 2s + 0.5s^2 + 0.5s}{s(s+1)(s+2)} = \frac{1}{s(s+1)(s+2)} \text{ OK!}$$

Then:

$$Y(s) = \frac{0.5}{s} - \frac{1}{(s+1)} + \frac{0.5}{(s+2)}$$
$$y(t) = \varepsilon(t)\left(0.5 - e^{-t} + 0.5e^{-2t}\right)$$

B) For random signal, first we consider what happens with the mean

$$\begin{aligned} h_1(t) &= \left(e^{-t} - e^{-2t}\right)\varepsilon(t) \\ y(t) &= x(t) * h_1(t) \\ \mu_y(t) &= E\{x(t) * h_1(t)\} = E\{x(t)\} * h_1(t) = \mu_x(t) * h_1(t) \end{aligned}$$

since

$$\mu_x(t) = \mu_x$$

(time independent)

$$\mu_y = \mu_x \int_{-\infty}^{\infty} h_1(t)\mathrm{d}t = \mu_x \int_{-\infty}^{\infty} \left(e^{-t} - e^{-2t}\right)\varepsilon(t)\mathrm{d}t = \mu_x \int_{0}^{\infty} \left(e^{-t} - e^{-2t}\right)\mathrm{d}t = 0.5\mu_x$$

For the ACF:

$$\varphi_{yy}(\tau) = \varphi_{hh}(\tau) * \varphi_{xx}(\tau)$$
$$\varphi_{hh}(\tau) = h_1(\tau) * h_1(-\tau)$$

Where $\varphi_{hh}(\tau)$ can be calculated from analytical expression for h_1.

Q2 (25p) Consider LTI system described by:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 5\frac{\mathrm{d}}{\mathrm{d}t} + 4\right] y(t) = \left[2\frac{\mathrm{d}}{\mathrm{d}t} + 6\right] x(t)$$

A) (10p) Find the impulse response $h(t)$

B) **(10p)** Find the unit step response $s(t)$ by using $\epsilon(t)$ as input

C) **(5p)** Verify your result by showing that $h(t) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)$

A) We will first find the system transfer function H(s) by taking the Laplace transform of the given ODE,

$$\left(\frac{d^2}{dt^2} + 5\frac{d}{dt} + 4\right) y(t) = \left(2\frac{d}{dt} + 6\right) x(t)$$
$$\Rightarrow (s^2 + 5s + 4)Y(s) = (2s + 6)X(s).$$

Transfer function and its partial fraction expansion (verify!) is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s+6}{s^2+5s+4} = \frac{2s+6}{(s+1)(s+4)} = \frac{4}{3}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s+4}$$

Thus the impulse response is

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \frac{2}{3}\left(2e^{-t} + e^{-4t}\right)\epsilon(t)$$

B) Laplace transform of the unit step input is $X(s) = 1/s$. The output of the system in the Laplace domain is given by

$$Y(s) = H(s)X(s) = \frac{2s+6}{(s+1)(s+4)}\frac{1}{s} = \frac{3}{2}\frac{1}{s} - \frac{4}{3}\frac{1}{s+1} - \frac{1}{6}\frac{1}{s+4}$$

Inverse transforming gives the time-domain response,

$$y(t) = \left[\frac{3}{2} - \frac{4}{3}e^{-t} - \frac{1}{6}e^{-4t}\right]\epsilon(t)$$

C) For $t > 0$ we get (othewise we have to differentiate $\epsilon(t)$),

$$\frac{dy(t)}{dt} = \left(\frac{4}{3}e^{-t} + \frac{4}{6}e^{-4t}\right)\cdot 1 = \frac{2}{3}(2e^{-t} + e^{-4t}) = h(t)$$

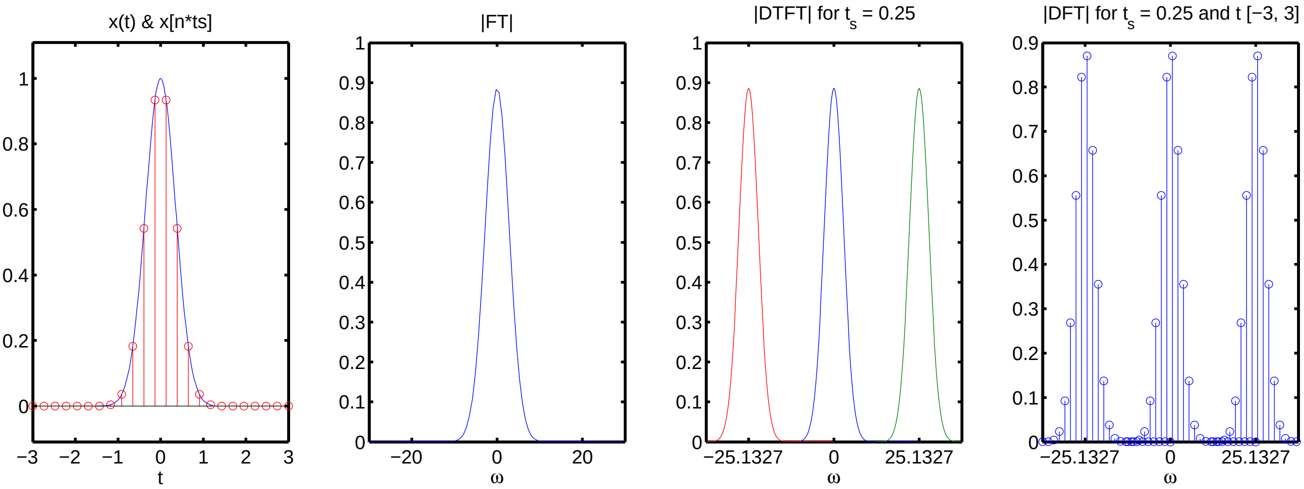
Q3 **(25p)** Explain the difference between FT, DFT and DTFT with respect to time domain signals for which those are calculated and resulting frequency representations. In a frequency range $\omega = \frac{2\pi}{f} \in [-30, 30]$, sketch approximate absolute values of FT, DTFT and DFT of a signal defined by:

$$y(t) = e^{-a^2t^2}$$

for $a = 2\mathrm{s}^{-1}$ and, where necessary, using sampling time $t_s = 0.25\mathrm{s}$ and signal duration $t \in [-3, 3]$.

HINT:

$$F_s(\omega) = 2\pi\omega_s \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$



Figur 1: Question Q3

FT

$$Y(j\omega) = \mathcal{F}\left\{e^{-a^2t^2}\right\} = \frac{\sqrt{\pi}}{a} e^{\frac{-\omega^2}{4a^2}}$$

$$F(0) = \frac{\sqrt{\pi}}{2} \quad F(4) = \frac{\sqrt{\pi}}{2} e^{-1} \quad F(8) = \frac{\sqrt{\pi}}{2} e^{-4}$$

and here x(t) is continius and defined for $t \in [-\infty, \infty]$. Resulting transform is continius in ω and also defined for $\omega \in [-\infty, \infty]$.

DTFT Here the time domain signal is discreet, $t \in [-\infty, \infty]$, the resulting transform is **continues in frequency**, and **periodic** with a period ω_s, $\omega \in [-\frac{\omega_s}{2}, \frac{\omega_s}{2}]$

$$\omega_s = \frac{2\pi}{t_s} = \frac{2\pi}{0.25} = 8\pi$$

$$F_s(\omega) = 2\pi\omega_s \sum_{n=-\infty}^{\infty} F(\omega - n\omega_s)$$

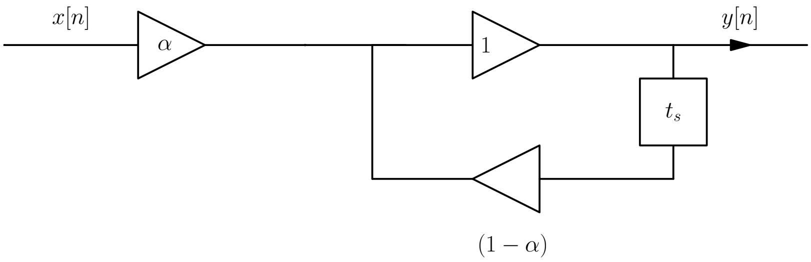
DFT Both **periodic** and **discreet** in time domain and in the frequency domain. To calculate we need to define time domain periodicity of the signal:

$$t = -3 : 3 \text{ s}$$
$$L = 6/0.25 = 24$$

DFT is periodic with a frequency $\omega = \omega_s$ and defined at points in the frequency space $\omega_s k/L$ where L is the length (periodicity) of the signal in the time domain.

Q4 (25p) The figure below (Figure 2) shows a digital filter (DSP) in which the delays are 0.5 ms.

A) (10p) Write down the difference equation and from this derive Z-transform of the transfer function. Using a method of choice, analyse the system and calculate 5 first output terms ($0 \le n < 5$) for the unit step excitation and $\alpha = 0.1535$. What kind of filter is this?



Figur 2: Question Q4

B) (15p) Now you would like to design an analogue 1st order Butterworth filter (using one capacitance C and one resistance R = 1000Ω) with approximately the same frequency response. Determine needed capacitance C and plot filter circuit diagram.

HINT Derive expression for inpulse response of the DSP and Butterworth filters. For filters with similar frequency response, the inpulse respons function will depend on time in the same manner ($h(t)/h(0) = h[n]/h[0]$). This also maight be useful:

$$\begin{aligned} a^n &= \left(e^{\beta}\right)^n \quad \ln a = \beta \\ t &= n \cdot t_s \end{aligned}$$

A)

For our filter we can write the difference equation as:

$$\begin{aligned} y[n] &= \alpha x[n] + (1-\alpha)y[n-1] \\ Y(z) &= \alpha X(z) + (1-\alpha)Y(z)z^{-1} \\ H(z) &= \frac{\alpha z}{z-(1-\alpha)} \end{aligned}$$

For unit step response:

$$\begin{aligned} X(z) &= \frac{z}{z-1} \\ Y(z) &= H(z)X(z) = \frac{\alpha z}{z-(1-\alpha)} \cdot \frac{z}{z-1} \end{aligned}$$

and then

$$y[n] = 1-(1-\alpha)^{n+1} \quad n \geq 0$$

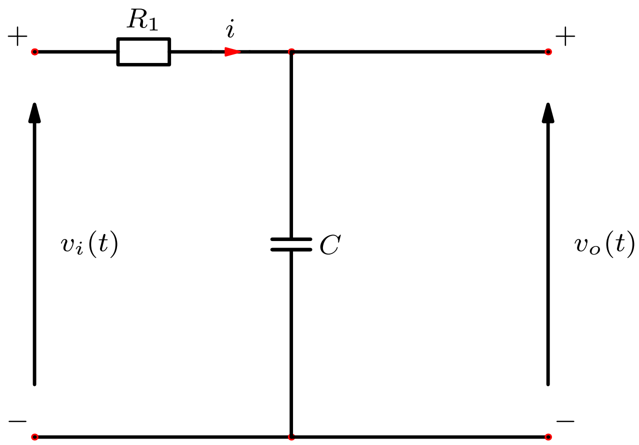
y[n] = [0.1535 0.2834 0.3934 0.4865 0.5654]

B)

DSP filters inpulse response can be calculated directly from its z-transform

$$\begin{aligned} H(z) &= \frac{\alpha z}{z-(1-\alpha)} = \alpha \frac{z}{z-(1-\alpha)} \\ h[n] &= \alpha(1-\alpha)^n \quad n \geq 0 \\ h[n] &= \alpha(1-\alpha)^n = \alpha\left(e^{\beta}\right)^n \quad 1-\alpha = e^{\beta} \\ \beta &= \ln(1-\alpha) \\ h[n] &= \alpha e^{\ln(1-\alpha)n} = 0.1535e^{-0.1667n} \end{aligned}$$

For $\alpha = 0.1535$, $\beta = -0.1667$. Now we need impulse response for a first order Butterworth low pass filter.



Figur 3: Circuit diagram for Butterworth low pass filter

$$
\begin{aligned}
H(s) &= \frac{1}{1+sRC} = \frac{1/RC}{s+\frac{1}{RC}} = \frac{\omega_c}{s+\omega_c} \\
h(t) &= \omega_c e^{-t\omega_c}
\end{aligned}
$$

Now we just have to compare time constant in both impulse response functions

$$
\begin{aligned}
h(t) &= \omega_c e^{-t\omega_c} \\
h[n] &= \alpha e^{\ln(1-\alpha)n}
\end{aligned}
$$

At $t = 1 \cdot t_s$, $n = 1$ and:

$$
\begin{aligned}
h(t_s) &= \omega_c e^{-t_s\omega_c} \\
h[1] &= a e^{\ln(1-\alpha)1} \\
t_s\omega_c &= -\ln(1-\alpha) \\
\frac{t_s}{RC} &= -\ln(1-\alpha) \\
\frac{1}{RC} &= \frac{-\ln(1-\alpha)}{t_s} \\
RC &= \frac{t_s}{-\ln(1-\alpha)} \\
C &= \frac{t_s}{-\ln(1-\alpha)R} = \frac{0.5\times 10^{-3}}{.167\cdot 1000} \\
C &= 3\times 10^{-6}\text{F}
\end{aligned}
$$

or simpler:

$$
\begin{aligned}
\frac{h(t)}{h(0)} &= \frac{h[n]}{h[0]} \\
e^{\ln(1-\alpha)n} &= e^{-t\omega_c} \\
\ln(1-\alpha) &= -\frac{t}{n}\omega_c = t_s\omega_c \\
-\frac{\ln(1-\alpha)}{t_s} &= \omega_c
\end{aligned}
$$

Appendix B.1 Bilateral Laplace Transform Pairs

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
$\delta(t)$	1	$s \in \mathbb{C}$
$\varepsilon(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\varepsilon(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$-e^{-at}\varepsilon(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < \mathrm{Re}\{-a\}$
$t\varepsilon(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
$t^n\varepsilon(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
$te^{-at}\varepsilon(t)$	$\frac{1}{(s+a)^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t^n e^{-at}\varepsilon(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$\cos(\omega_0 t)\varepsilon(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
$e^{-at}\cos(\omega_0 t)\varepsilon(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$e^{-at}\sin(\omega_0 t)\varepsilon(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > \mathrm{Re}\{-a\}$
$t\cos(\omega_0 t)\varepsilon(t)$	$\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$
$t\sin(\omega_0 t)\varepsilon(t)$	$\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$	$\mathrm{Re}\{s\} > 0$

Appendix B.2 Properties of the Bilateral Laplace Transform

$x(t)$	$X(s) = \mathcal{L}\{x(t)\}$	ROC
Linearity $Ax_1(t) + Bx_2(t)$	$AX_1(s) + BX_2(s)$	ROC $\supseteq$ ROC$\{X_1\}$ $\cap$ROC$\{X_2\}$
Delay $x(t-\tau)$	$e^{-s\tau}X(s)$	not affected
Modulation $e^{at}x(t)$	$X(s-a)$	Re$\{a\}$ shifted by Re$\{a\}$ to the right
'Multiplication by t', Differentiation in the frequency domain $tx(t)$	$-\frac{d}{ds}X(s)$	not affected
Differentiation in the time domain $\frac{d}{dt}x(t)$	$sX(s)$	ROC $\supseteq$ ROC$\{X\}$
Integration $\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	ROC $\supseteq$ ROC$\{X\}$ $\cap\{s : \mathrm{Re}\{s\} > 0\}$
Scaling $x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	ROC scaled by a factor of a

Appendix B.3 Fourier Transform Pairs

$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\dot{\delta}(t)$	$j\omega$
$\frac{1}{T}\perp\!\!\perp\!\!\perp\left(\frac{t}{T}\right)$	$\perp\!\!\perp\!\!\perp\left(\frac{\omega T}{2\pi}\right)$
$\varepsilon(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\mathrm{rect}(at)$	$\frac{1}{\lvert a\rvert}\mathrm{si}\left(\frac{\omega}{2a}\right)$
$\mathrm{si}(at)$	$\frac{\pi}{\lvert a\rvert}\mathrm{rect}\left(\frac{\omega}{2a}\right)$
$\frac{1}{t}$	$-j\pi\mathrm{sign}(\omega)$
$\mathrm{sign}(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin(\omega_0 t)$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
$e^{-\alpha\lvert t\rvert},\ \alpha > 0$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-a^2t^2}$	$\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$

Appendix B.4 Properties of the Fourier Transform

	$x(t)$	$X(j\omega) = \mathcal{F}\{x(t)\}$
Linearity	$Ax_1(t) + Bx_2(t)$	$AX_1(j\omega) + BX_2(j\omega)$
Delay	$x(t-\tau)$	$e^{-j\omega\tau}X(j\omega)$
Modulation	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
'Multiplication by t' Differentiation in the frequency domain	$tx(t)$	$-\frac{dX(j\omega)}{d(j\omega)}$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$X(j\omega)\left[\pi\delta(\omega)+\frac{1}{j\omega}\right]$ $= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right),\quad a \in \mathbb{R}\backslash\{0\}$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)\cdot X_2(j\omega)$
Multiplication	$x_1(t)\cdot x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Duality	$x_1(t)$ $x_2(jt)$	$x_2(j\omega)$ $2\pi x_1(-\omega)$
Symmetry relations	$x(-t)$ $x^*(t)$ $x^*(-t)$	$X(-j\omega)$ $X^*(-j\omega)$ $X^*(j\omega)$
Parseval theorem	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(j\omega)\rvert^2 d\omega$

Appendix B.6 Properties of the z-Transform

Property	$x[k]$	$X(z)$	ROC
Linearity	$ax_1[k]+bx_2[k]$	$aX_1(z)+bX_2(z)$	ROC $\supseteq$ ROC$\{X_1\}\cap$ROC$\{X_2\}$
Delay	$x[k-\kappa]$	$z^{-\kappa}X(z)$	ROC$\{x\}$; separate consideration of $z=0$ and $z\to\infty$
Modulation	$a^k x[k]$	$X\left(\frac{z}{a}\right)$	ROC$=\left\{z\,\middle\|\,\frac{z}{a}\in\text{ROC}\{x\}\right\}$
Multiplication by k	$kx[k]$	$-z\frac{dX(z)}{dz}$	ROC$\{x\}$; separate consideration of $z=0$
Time inversion	$x[-k]$	$X(z^{-1})$	ROC$=\{z\,\|\,z^{-1}\in\text{ROC}\{x\}\}$
Convolution	$x_1[k]*x_2[k]$	$X_1(z)\cdot X_2(z)$	ROC $\supseteq$ ROC$\{x_1\}\cap$ROC$\{x_2\}$
Multiplication	$x_1[k]\cdot x_2[k]$	$\frac{1}{2\pi j}\oint X_1(\zeta)X_2\left(\frac{z}{\zeta}\right)\frac{1}{\zeta}d\zeta$	multiply the limits of the ROC

Appendix B.5 Two-sided z-Transform Pairs

$x[k]$	$X(z)=\mathcal{Z}\{x[k]\}$	ROC
$\delta[k]$	1	$z\in\mathbb{C}$
$\varepsilon[k]$	$\frac{z}{z-1}$	$\|z\|>1$
$a^k\varepsilon[k]$	$\frac{z}{z-a}$	$\|z\|>\|a\|$
$-a^k\varepsilon[-k-1]$	$\frac{z}{z-a}$	$\|z\|<\|a\|$
$k\varepsilon[k]$	$\frac{z}{(z-1)^2}$	$\|z\|>1$
$ka^k\varepsilon[k]$	$\frac{az}{(z-a)^2}$	$\|z\|>\|a\|$
$\sin(\Omega_0 k)\varepsilon[k]$	$\frac{z\sin\Omega_0}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$
$\cos(\Omega_0 k)\varepsilon[k]$	$\frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$	$\|z\|>1$