Solution set for final exam in TFY 4305

December 12, 2009

Problem 1

a) $\dot{x} = 0 = y(x+1) \Rightarrow x = -1,\ y = 0$

$\dot{y} = 0 = x(1+y^3) \Rightarrow x = 0,\ y = -1$

$$\Rightarrow \quad \underline{\underline{(x^*, y^*) = (0,0) \text{ or } (-1,-1)}}$$

b)

Jacobi matrix: $J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x+1 \\ 1+y^3 & 3xy^2 \end{pmatrix}$

At $(0,0)$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} \tau = 0 = \lambda_1 + \lambda_2 \\ \Delta = -1 = \lambda_1 \cdot \lambda_2 \end{array}\right\}$$

$$\Rightarrow \quad \lambda = \pm 1 \Rightarrow \underline{\text{saddle point}}$$

At $(-1,-1)$:

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\tau = \lambda_1 + \lambda_2 = -4$$

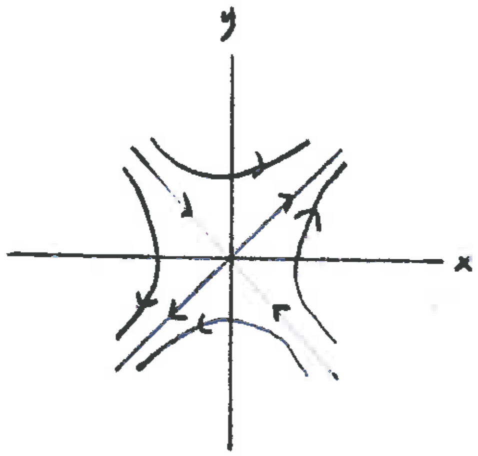
$$\Delta = \lambda_1 \cdot \lambda_2 = 3$$

$$\Rightarrow \lambda_1 = -1, \quad \lambda_2 = -3$$

<u>Stable node.</u>

c) Eigendirections for $(0,0)$

$$J\vec{v} = \lambda \vec{v} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

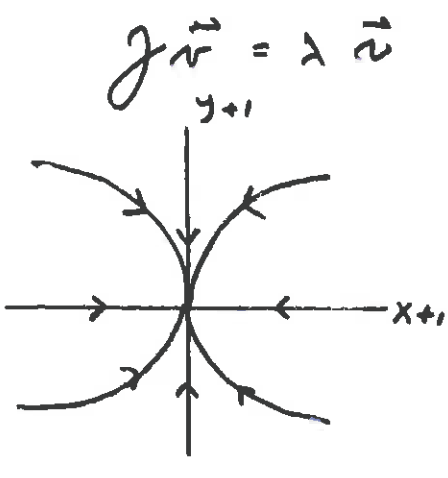


$$\Rightarrow \lambda = 1; \quad \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -1; \quad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigendirections for $(-1,-1)$

$$J\vec{v} = \lambda \vec{v} \qquad \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$



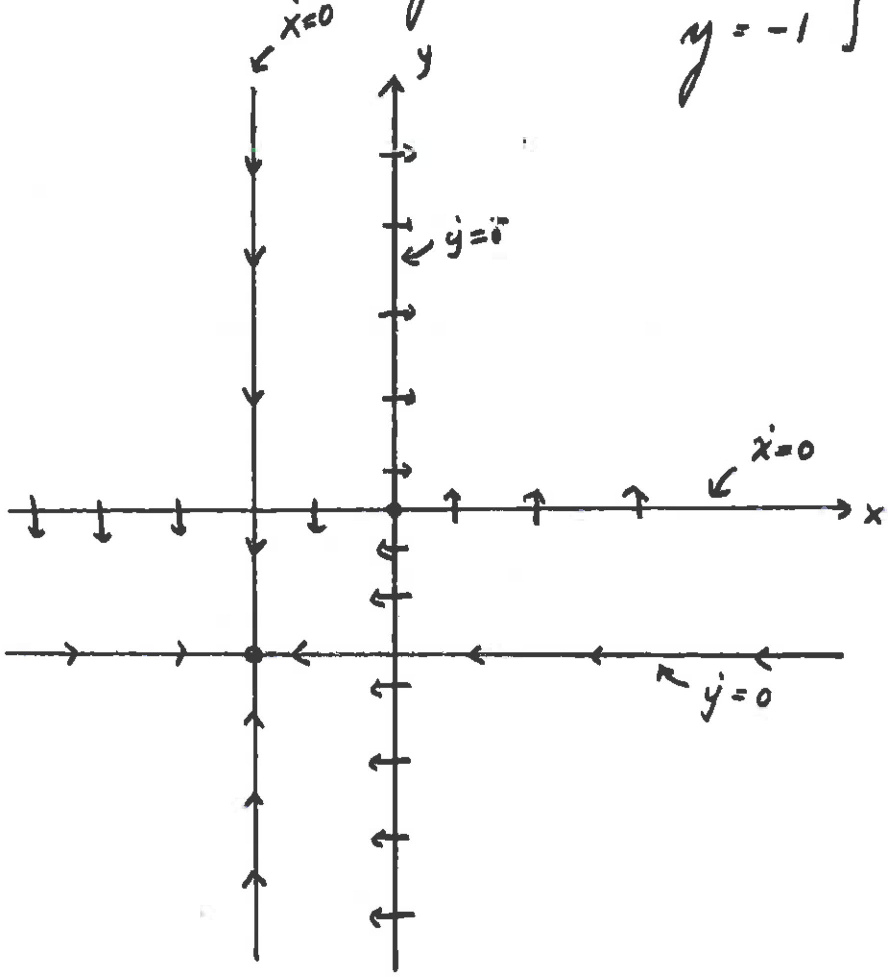
$$\Rightarrow \lambda = -1; \quad \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda = -3; \quad \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>Nullclines :</u>

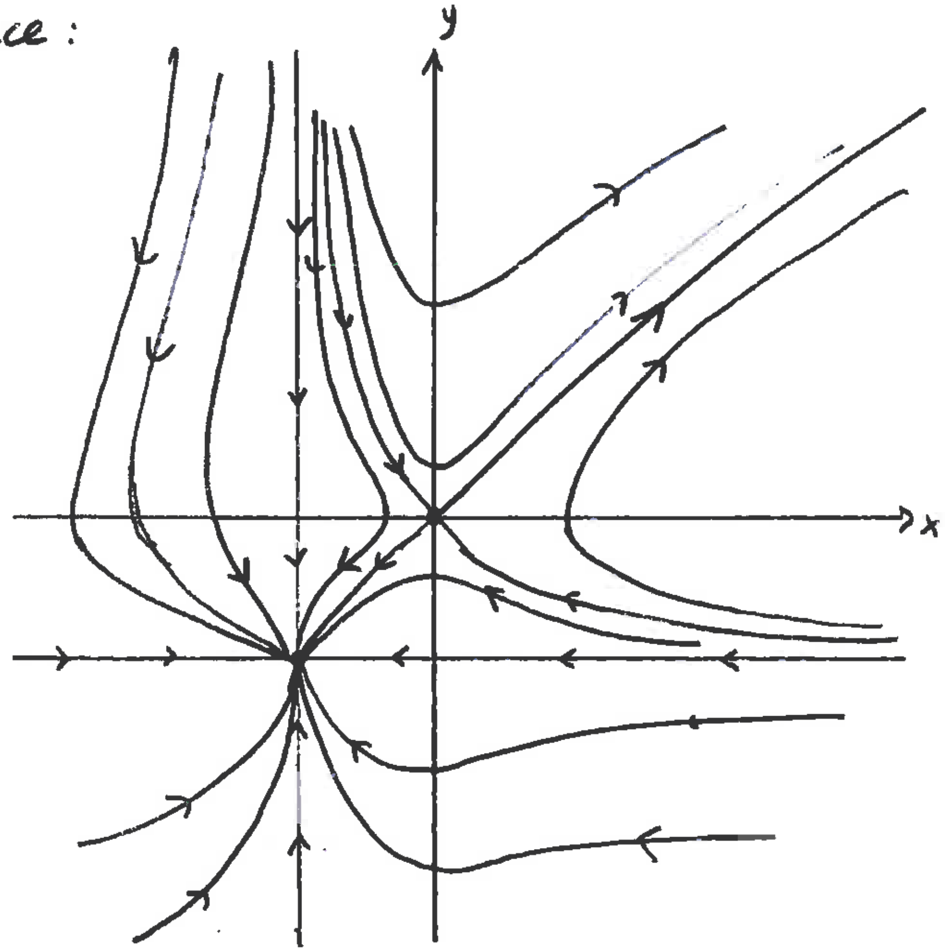
$$\dot{x} = 0 \Rightarrow \left.\begin{array}{l} y = 0 \\ x = -1 \end{array}\right\}$$

$$\dot{y} = 0 \Rightarrow \left.\begin{array}{l} x = 0 \\ y = -1 \end{array}\right\}$$



The signs have been placed using the flows near the two fixed points.

We now combine all the information into a complete sketch of phase space:



Problem 2

a) $\begin{rcases} x = -2y^2 \\ xy = x^2 y \end{rcases}$ We see immediately that $(x, y) = (0,0)$ is a solution.

Is it the only one?

Suppose $x \neq 0$. Then $x = -2y^2 \Rightarrow y \neq 0$

$$xy = x^2 y \Rightarrow \left.\begin{array}{l} x = 1 \\ y^2 = -\frac{1}{2} \end{array}\right\}$$

$\Rightarrow$ No solution.

Suppose $y \neq 0$. Then $-2y^2 = x \Rightarrow x \neq 0$;

and there is no solution.

$\Rightarrow$ <u>(0,0) is the only fixed point.</u>

b) To have a Liapunov function, it must satisfy

① $V(x^*, y^*) = 0$

② $V(x, y) > 0 \qquad (x, y) \neq (x^*, y^*)$

③ $\dot{V}(x, y) < 0 \qquad (x, y) \neq (x^*, y^*)$.

Using the proposed function, ① and ②
are immediately fulfilled.

We now investigate ③:

$$\frac{dV}{dt} = \frac{d}{dt}\{x^2 + ay^2\} = 2x\dot{x} + 2ay\dot{y}$$

$$= 2x\{-x - 2y^2\} + 2ay\{xy - x^2y\}$$

$$= -2x^2 - 4xy^2 + 2axy^2 - 2ax^2y^2$$

If we now set $a = 2$, we get

$$\frac{dV}{dt} = -2x^2 - 4x^2y^2 < 0 \quad \text{when } (x,y) \neq (0,0).$$

Hence, $\underline{\underline{V(x,y) = x^2 + 2y^2}}$ is a
Liapunov function and the fixed
point is stable.

Problem 3

a)

$$\frac{\partial H}{\partial \phi_n} = \frac{\partial}{\partial \phi_n}\left\{ \frac{1}{2}(\phi_{n+1}-\phi_n)^2 + \frac{1}{2}(\phi_n-\phi_{n-1})^2 \right.$$

$$\left. + \frac{a}{4}(\phi_n^2-1)^2 \right\}$$

These are the only terms in H containing ϕ_n.

$$\frac{\partial H}{\partial \phi_n} = -(\phi_{n+1}-\phi_n) + (\phi_n - \phi_{n-1})$$

$$+ \frac{a}{2}(\phi_n^2-1)2\phi_n = 0$$

$\Longrightarrow$

$$\underline{\underline{\phi_{n+1}+\phi_{n-1}-2\phi_n = a\phi_n(\phi_n^2-1)}}$$

b)

Set $x_n = \phi_n$ and

$y_n = \phi_{n-1}$

Then, the expression here may be written

$$x_{n+1} + y_n - 2x_n = a x_n (x_n^2 - 1)$$

$\Longrightarrow$

$$\underline{x_{n+1} = 2x_n + a x_n (x_n^2 - 1) - y_n}$$

From $y_n = \phi_{n-1}$, we have that

$$\underline{y_{n+1}} = \phi_n \; \underline{= x_n}$$

Hence,

$$\underline{\underline{\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2x_n + a x_n (x_n^2 - 1) - y_n \\ x_n \end{pmatrix}}}$$

$$= T \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

c) We consider the Jacobian of T:

$$J = \begin{pmatrix} \dfrac{\partial x_{n+1}}{\partial x_n} & \dfrac{\partial x_{n+1}}{\partial y_n} \\ \dfrac{\partial y_{n+1}}{\partial x_n} & \dfrac{\partial y_{n+1}}{\partial y_n} \end{pmatrix} = \begin{pmatrix} 2-a+3ax_n^2 & -1 \\ 1 & 0 \end{pmatrix}$$

The determinant is:

$$\Delta = 1 \implies \underline{\text{Area preserving.}}$$

d) The fixed point equation is:

$$\left.\begin{aligned} x^* &= 2x^* + ax^*(x^{*2}-1) - y^* \\ y^* &= x^* \end{aligned}\right\} \implies$$

$$ax^{*3} = ax^* \implies x^* = 0, \pm 1$$

$$\implies y^* = 0, \pm 1$$

$$\underline{(x^*, y^*) = (0,0), (1,1), (-1,-1)}$$

e) $(0,0)$:

$$\begin{vmatrix} 2-a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(2-a-\lambda)+1=0$$

$$\lambda^2-(2-a)\lambda+1=0$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\left\{2-a \pm \sqrt{a^2-4a}\right\}$$

When $a<4$, they are complex numbers on the unit circle.

$(1,1)$ and $(-1,-1)$:

$$\begin{vmatrix} 2+2a-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = -\lambda(2+2a-\lambda)+1=0$$

$$\lambda^2-2(1+a)\lambda+1=0$$

$$\Rightarrow \quad \lambda_\pm = 1+a \pm \sqrt{2a+a^2}$$

$\lambda_+ > 1$: Unstable eigendirection for all a.

$\lambda_- < 1$: stable eigendirection for all a.

f) When $a \to 4$, the two complex eigenvalues collide. $\lambda = -1$ at this point. We are therefore dealing with a period doubling at this point.