Norwegian University of Science and Technology
Department of Physics

Contact: Jacob Linder
Phone: 735 918 68

Suggested solution for Exam TFY4345: Classical Mechanics

PROBLEM 1

(a) The corresponding force to this potential is given by $\mathbf{F} = -\nabla V$, such that $\mathbf{F} = F\hat{\mathbf{z}}$. This is a constant force in the z-direction, such that the particle will be accelerated from rest in the positive z-direction.

(b) The Lagrangian is given by $L = m\dot{z}^2/2 + Fz$. The extremal values of the action

$$I = \int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t \tag{1}$$

are provided by the Lagrange-equation. This gives us $m\ddot{z} = F$. Inserting our ansatz for $z(t)$, we obtain $C = F/(2m)$. From the conditions $z(t=0) = 0$ and $z(t=t_0) = a$, we find $A = 0$ and $B = [a - Ft_0^2/(2m)]/t_0$.

(c) The variation of the action in this case reads:

$$\delta I = \int_{t_1}^{t_2} \mathrm{d}t \Big[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta\ddot{q}\Big] \tag{2}$$

Now use that $\delta\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\delta q$ and similarly for $\ddot{q}$. We then obtain by means of a partial integration:

$$\begin{aligned}\delta I = &\int_{t_1}^{t_2} \mathrm{d}t \Big[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}\delta\dot{q}\Big] \\ &+ \Big|_{t_1}^{t_2}\delta q\frac{\partial L}{\partial \dot{q}} + \Big|_{t_1}^{t_2}\delta\dot{q}\frac{\partial L}{\partial \ddot{q}}.\end{aligned} \tag{3}$$

The two last terms vanish since there is no variation at the end-points t_1 and t_2. Apply then one more partial integration to the third term in the above equation to arrive at the final result.

(d) The corresponding equation of motion reads $m\ddot{q}/2 + kq = 0$, which describes a harmonic oscillator.

(e) If a Lagrangian has a continuous symmetry, for instance invariance under time-translation $t \to t + \Delta t$ or space-translation $r \to r + \Delta r$, it also has a belonging conserved quantity. In the specific example given in the exam, the Lagrangian has no explicit dependence on time and hence energy is conserved. The Lagrangian is, however, not invariant under a space-translation, and hence the total momentum cannot be conserved.

PROBLEM 2

(a) The corresponding potential to the written force reads $V = -k/r$, and so the Lagrangian reads $L = (m/2)(\dot{r}^2 + r^2\dot{\theta}^2) + k/r$. It follows that θ is a cyclic coordinate, such that the canonical momentum $p_\theta = mr^2\dot{\theta}$ is a conserved quantity. This may be identified as the angular momentum of the particle since we have a complete radial symmetry in the problem, i.e. $l = mr^2\dot{\theta}$. Using this, we obtain the following Lagrange-equation for the coordinate r:

$$m\ddot{r} = -k/r^2 + l^2/(mr^3). \tag{4}$$

It follows from the expression for energy $E = T + V$ that the effective potential may then be written as $V'(r) = V(r) + l^2/(2mr^2)$. The discussion of the qualitative nature of the motion of this particle depending on energy E and angular momentum l is given in the compendium on p. 38-39.

(b) The differential scattering cross section expresses the ratio of the number of particles scattered into a specific part of space per time and the intensity of the incident particles. The total scattering cross section is a measure for the effective scattering area that incident particles will be subject to.

(c) This means that any incident particles towards a Coulomb-potential will be scattered, regardless of how large their impact parameter is.

(d) The effective scattering cross section will be πa^2. The reason for this is that particles incident within this area will hit the sphere and be scattered, whereas particles outside of this area will remain unaffected.

(e) The system is in equilibrium when all the generalized forces vanish, $Q_i = -\frac{\partial V}{\partial q_i}\Big|_0 = 0$, where the subscript $_0$ indicates the equilibrium values of the coordinates $\{q_i\}$, in effect $q_1 = q_{0,1}$, $q_2 = q_{0,2}$, ... Upon a slight perturbation of the coordinates from their equilibrium values, the system can either undergo a bound or unbound movement depending on whether the equilibrium is stable or unstable. For a stable equilibrium, the vanishing of the generalized forces indicate that the potential is at a minimum. For an unstable equilibrium, the vanishing of the generalized forces indicate that the potential is at a maximum.

PROBLEM 3

(a) In the context of particle collisions, the threshold energy is the minimum kinetic energy required to enable a reaction.

(b) The final state with new particles produced will have the lowest energy if all particles are at rest. In order to satisfy conservation of momentum, this means that the total momentum of the original colliding particles must be zero. Hence, the collision must take place in the center-of-mass system. If the original particles have a finite total momentum, the produced particles must also have a finite total momentum to satisfy conservation of momentum, which means that extra kinetic energy is needed originally to give the produced particles their necessary momentum.

(c) Conservation of energy and momentum is expressed via conservation of 4-momemtum p_μ. We also know that the product $p_\mu p_\mu$ is a Lorentz-invariant, in effect it's the same in any inertial system. Let p_μ denote the total 4-momentum of the system before the collision whereas p'_μ denotes the total 4-momentum after the collision. We may define the equivalent mass of the system M as:

$$p_\mu p_\mu = p'_\mu p'_\mu = -M^2c^2. \tag{5}$$

In general, this mass is not equal to the sum of the masses before and after the collision, respectively. However, if all the particles after the collision are at rest, there is no kinetic energy and in this case we have $M = M_{\text{tot}}$ where M_{tot} is the total rest mass of the particles produced in the collision.

Let's now analyze the product $p_\mu p_\mu$ first in the system where one particle initially is at rest and then in the COM system. In the first case, we obtain:

$$p_\mu p_\mu = -m_1^2c^2 - m_2^2c^2 + 2(\mathbf{p_1}\cdot\mathbf{p_2} - E_1E_2/c^2). \tag{6}$$

Now use that one particle is at rest (set for instance $\mathbf{p_2} = 0$) and that the kinetic energy of the other particle (for instance particle 1) then reads $K_1 = E_1 - m_1c^2$. The above equation can then be rewritten as:

$$M^2c^4 = (m_1+m_2)^2c^4 + 2m_2c^2K_1 \tag{7}$$

which gives us the result for the total kinetic energy of the system before the collision:

$$K_1 = \frac{M^2c^4 - (m_1+m_2)^2c^4}{2m_2c^2}. \tag{8}$$

Consider now the COM system. Since the total momentum before the collision is now zero, we obtain from the definition of p_μ:

$$M^2c^2 = E^2_{\text{before}}/c^2 \tag{9}$$

The total energy before the collision is the sum of the kinetic and rest energies of the two particles, in effect $E_{\text{before}} = K_1 + K_2 + m_1c^2 + m_2c^2$. We can then identify the result for the total kinetic energy of the system before the collision:

$$K_1 + K_2 = Mc^2 - (m_1+m_2)c^2 \tag{10}$$

In order to now say something about the threshold energies in the two scenarios above, we use the fact that the threshold energy corresponds to the case where the final particles are at rest after the collision in the COM-system. In this case, we have $M = M_{\text{tot}}$. This also holds in the system where only one of the particles was at rest initially, since $p_\mu p_\mu = -M^2c^2$ is an invariant. According to the assumption written in the problem text, we may write $M_{\text{tot}} = k(m_1+m_2)$.

The remaining part of the problem is now to prove that the threshold energy in the COM system is lower than in the system where one particle was initially at rest. Using Eqs. (8) and (10), we must check if

$$M_{\text{tot}}c^2 - (m_1+m_2)c^2 \leq \frac{M^2_{\text{tot}}c^4 - (m_1+m_2)^2c^4}{2m_2c^2} \tag{11}$$

holds. Inserting $M_{\text{tot}} = k(m_1+m_2)$ gives us the inequality:

$$m_1^2(k^2-1) + 2m_1m_2(k^2-k) + m_2^2(k^2-2k+1) \geq 0. \tag{12}$$

Since the masses $\{m_1, m_2\}$ must be larger than zero and we know that $k \geq 1$, the first two terms are guaranteed to be positive. The last term is also guaranteed to be positive since $k^2 - 2k + 1 \geq 0$. Therefore, the entire left-hand side of the equation must be positive and the inquality is satisfied which proves our statement.

PROBLEM 4

(a) The key here is to use a proper coordinate system for (B) and (C) so that we can use the Lorentz-transformation listed in the Supplementary Information. Let's first consider situation (B) and choose the velocity direction to be the z-axis. In that case, we have $\mathbf{E} = -E\hat{\mathbf{z}}$ for the observer in $\mathcal{S}$ [or equivalently for the capacitor's reference system in (B)] since the field points from positive to negative charge. The corresponding electric potential is then $\phi = Ez$ such that the 4-vector A_μ becomes $A_\mu = (0, \mathrm{i}E/c)$. Using the general form for $F_{\mu\nu}$, we thus obtain only two non-zero elements for the electromagnetic field tensor: $F_{34} = -F_{43} = \mathrm{i}E/c$.

We are now in a position to evaluate the field seen by the stationary observer in $\mathcal{S}$. The Lorentz transformation gives us:

$$F'_{\mu\nu} = (L_{\mu 3}L_{\nu 4} - L_{\mu 4}L_{\nu 3})F_{34}. \tag{13}$$

We only obtain two non-zero elements for $F'_{\mu\nu}$ as well, namely $F'_{34} = F_{34}$ and $F'_{43} = F_{43}$. Therefore, the electric field seen by the observer in (B) is the same as in (A). This reflects the fact that the component of the electric field along the velocity direction remains invariant under a Lorentz transformation.

We now perform the equivalent analysis for situation (C). In this case, it is useful to choose the coordinate system so that the velocity direction again is the z-axis. The electric field seen from the capacitor's reference system is then $\mathbf{E} = E\hat{\mathbf{x}}$. Performing the same analysis as above, we find that

$$F'_{13} = \mathrm{i}\beta\gamma F_{14},\ F'_{14} = \gamma F_{14}. \tag{14}$$

Considering the general expression for $F_{\mu\nu}$, we can infer that the above equations are equivalent to:

$$B'_y = -\beta\gamma E/c,\ E'_x = \gamma E_x. \tag{15}$$

In effect, a stationary observer in $\mathcal{S}$ will see a modified electric field and additionally a magnetic field along the y-axis.

(b) In situation (B), there is a contraction of the distance d between the plates, but no contraction of the area of the plates since the velocity is perpendicular to the plates. Since the electric field is independent of d, it remains invariant. However, in situation (C), the velocity is parallell with the plates, so that a stationary observer will see a contraction of the area of the plates: $A \to A/\gamma$. Inserting this into the expression for the electric field, we find that $E \to \gamma E$, which is precisely the result obtained in **(a)**.

(c) Exactly this derivation is shown in the compendium in p. 109.