Norwegian University of Science and Technology
Department of Physics

Contact during exam: Jacob Linder
Phone: 735 918 68

Exam TFY4345: Classical Mechanics
Friday June 1st 2012
09.00-13.00
English

The exam consists of 4 problems. Each problem counts for in total 25% of the total weight of the exam, but each sub-exercise **(a)**, **(b)**, etc. does not necessarily count equally.

Read each problem carefully in order to avoid unnecessary mistakes.

Allowed material to use at exam: C.

- Approved, simple calculator.
- K. Rottmann: Matematisk formelsamling.
- K. Rottmann: Mathematische Formelsammlung. Barnett & Cronin: Mathematical Formulae.

Also consider the Supplementary Material on the last page of this exam.

PROBLEM 1

(a) Two point masses m are joined by a rigid weightless rod of length l, the center of which is constrained to move on a circle of radius r_0. Assume that the masses are also restricted to move in the plane defined by the motion of the center of the rod. Derive the total kinetic energy of this system expressed in terms of two generalized coordinates.

(b) Define in words, and in detail, what the differential scattering cross section gives information about physically. Also define in words the difference between the differential scattering cross section and the total scattering cross section.

(c) Explain in detail the relation between symmetries of the Lagrangian describing a given system and the possibility of having conserved quantities. Give at least three examples of important symmetries a Lagrangian can have and the corresponding conserved physical quantities.

(d) Describe how frictional forces may be included in Lagrange's equations via the Rayleigh dissipation function and provide the typical form for this function.

(e) Define in words the concept of ”threshold energy” in the context of particle collisions.

(f) Explain in detail the meaning of the following concepts in relativity:

- Length contraction.
- Time dilation.

Finally, explain in detail the concept of gauge-invariance in the context of electromagnetic fields.

PROBLEM 2

(a) A particle of mass m is restricted to move under the influence of gravity, but without friction, on the inside of a paraboloid described by the equation $z = \alpha r^2$. Here, r is the radial vector in the xy-plane. Find the one-dimensional problem equivalent to its motion, i.e. write down the corresponding Lagrangian and its equations of motion. Assume now that the particle moves along a circle, and find an analytical expression for the angular velocity of this circular motion.

(b) Newton's second law may be written as $\dot{\mathbf{p}} = f(r)\mathbf{r}/r$ for a general central force. Assume that $f(r)$ is proportional to r^{-2}, and show that the Laplace-Runge-Lenz vector $\mathbf{A}$ is conserved. This vector is defined as:

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\mathbf{r}/r \tag{1}$$

where $\mathbf{L}$ is the angular momentum, m is the mass, and k is the constant of proportionality between $f(r)$ and r^{-2}, i.e. $f(r) = kr^{-2}$.

(c) Consider the operator equation relating the time-change of a vector as seen in a fixed, stationary system and as seen in a (body) system rotating with angular velocity ω:

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + \omega\times \tag{2}$$

Assume that the origo of the two reference-systems coincide. Use this equation on the position vector $\mathbf{r}$ to an object with mass m in order to derive an expression for the effective force experienced by this object in the rotating system expressed with three ingredients: 1) the force acting on the object as seen from the stationary system, 2) the Coriolis-force, and 3) the centrifugal force. Identify a concrete analytical expression for the Coriolis and centrifugal force.

PROBLEM 3

(a) Consider a rigid body rotating around its own center of mass with an angular velocity ω. Show that the total angular momentum of the rigid body may be written as:

$$\mathbf{L} = I\omega \tag{3}$$

where I is the moment of inertia matrix, and write down a general expression for the components I_{ij} of this matrix.

(b) Consider the CO_2 molecule as a linear, tri-atomic molecule positioned along the x-axis. Assume that the masses of the oxygen atoms are m while the mass of the carbon atom is M. Moreover, assume that the distance between the C and O atoms in equilibrium is d. Consider now a situation where we perturb slightly the equilibrium situation such that the position of the atoms deviate from their equilibrium positions x_{0i}, where $i = 1,2,3$ refers to the right O, middle C, and left O atoms, respectively. The deviations may then be quantified as $\eta_i = x_i - x_{0i}$.

Write down the Lagrangian of the system expressed in terms of η_i. Assume now that the center of mass is at rest and introduce the so-called normal coordinates $Q_a = \eta_1 + \eta_3$ and $Q_s = \eta_1 - \eta_3$. Use this information to write the Lagrange-equations for the normal coordinates and identify the belonging eigenfrequencies ω_a and ω_s.

If we relaxed our assumption about a center of mass at rest, we would find one more possible solution for the eigenfrequency ω. Which one? Discuss the physical meaning of its value.

(c) A charged particle is constrained to move in the xy-plane under the influence of a central force potential $V = \frac{1}{2}kr^2$, which is non-electromagnetic, in addition to a constant magnetic field $\mathbf{B}$ perpendicular to the plane. In this case, the magnetic vector potential may be written $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$. Assume that the Lagrangian of the system may be written as:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\dot{\mathbf{x}} \cdot \mathbf{A}) - \frac{k}{2}(x^2 + y^2). \tag{4}$$

Switch to polar coordinates and write down the corresponding Hamilton-Jacobi equation based on this Lagrangian.

PROBLEM 4

(a) A meson of mass m_π at rest disintegrates into a meson of mass m_μ and a neutrino of zero mass. Derive an expression for the kinetic energy of the μ-meson.

(b) A photon may be described clasically as a particle of zero mass possessing nevertheless a momentum $h/\lambda = h\nu/c$, and therefore a kinetic energy $h\nu$, where h is Planck's constant, c is the speed of light, and ν is the frequency of the photon. Assume that the photon collides with an electron of mass m at rest and scatters to an angle θ with a new energy $h\nu'$. Show that the change in energy is related to the scattering angle by the formula:

$$\lambda' - \lambda = 2\lambda_c \sin^2(\theta/2), \tag{5}$$

where $\lambda_c = h/mc$ is the Compton wavelength. Also find an expression for the kinetic energy of the electron after the collision. Consider the special case of $\theta = 0$. How do you interpret the corresponding change in energy and the kinetic energy obtained by the electron?

Supplementary Information

The regime of validity and the meaning of the symbols below are assumed to be known by the reader.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{6}$$

$$[u,v]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \tag{7}$$

$$\begin{aligned} x_\mu &= (\mathbf{r}, \mathrm{i}ct), \\ p_\mu &= (\mathbf{p}, \mathrm{i}E/c) \end{aligned} \tag{8}$$

$$A_\mu = (\mathbf{A}, \mathrm{i}\phi/c),\ \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t,\ \mathbf{B} = \nabla\times\mathbf{A} \tag{9}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{10}$$

From the above equations, it follows that the general form of $F_{\mu\nu}$ in a given reference system is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\mathrm{i}E_x/c \\ -B_z & 0 & B_x & -\mathrm{i}E_y/c \\ B_y & -B_x & 0 & -\mathrm{i}E_z/c \\ \mathrm{i}E_x/c & \mathrm{i}E_y/c & \mathrm{i}E_z/c & 0 \end{pmatrix} \tag{11}$$

$$F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}. \tag{12}$$

The Lorentz-transformation matrix for the situation in Fig. 1 is given by:

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \mathrm{i}\beta\gamma \\ 0 & 0 & -\mathrm{i}\beta\gamma & \gamma \end{pmatrix} \tag{13}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

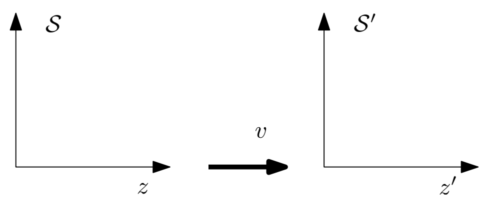

FIG. 1: Lorentz-transformation along the z-axis.