FY8201 / TFY8 **Nanoparticle and polymer physics I**
SOLUTION of EXERCISE 7

Eq. (x.x) refers to version AM24nov05 of lecture notes: "Nano-particle and polymer physics". Equations pertinent to this exercise you will find in Ch. 5.2.1

A) The sphere perturbs the velocity field with an amount $\vec{v}'$, and the resultant fluid velocity is thus

$$\vec{v} = \vec{u} + \vec{v}' \ , \tag{1}$$

where $\vec{u}$ is the stationary velocity field. We apply coordinates $(\vec{\delta}_r, \vec{\delta}_\theta, \vec{\delta}_\phi)$ as given in Fig. 5.2 of lecture notes, i.e. $\vec{v} = v_r\vec{\delta}_r + v_\theta\vec{\delta}_\theta + v_\phi\vec{\delta}_\phi$. Axial symmetry about z-axis implies $v_\phi = 0$. We choose to align the stationary velocity field with the z-axis: $\vec{u} = -u\vec{\delta}_z = -u\cos\theta\vec{\delta}_r + u\sin\theta\vec{\delta}_\theta$.

According to Eqs. (5.46)-(5.47) the resulting velocity field is expressed

$$v_r(r,\theta) = -u\left[1 - \frac{3}{2}\left(\frac{\sigma}{r}\right) + \frac{1}{2}\left(\frac{\sigma}{r}\right)^3\right]\cos\theta \tag{2}$$

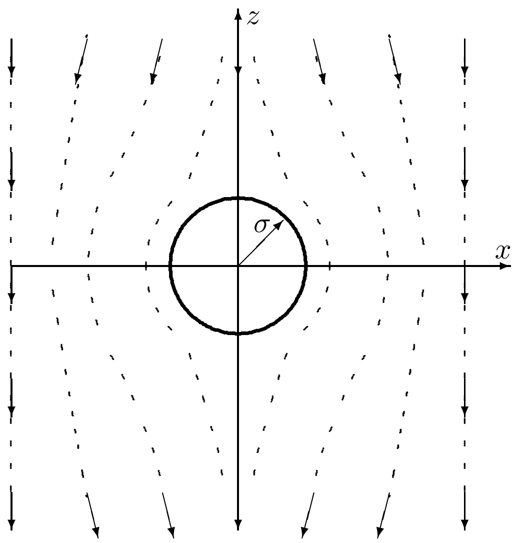
$$v_\theta(r,\theta) = u\left[1 - \frac{3}{4}\left(\frac{\sigma}{r}\right) - \frac{1}{4}\left(\frac{\sigma}{r}\right)^3\right]\sin\theta. \tag{3}$$

Note that the boundary conditions are fulfilled:

at the bead surface $r = \sigma$: $\quad v_\theta(\sigma,\theta) = 0$ (no slip condition) $\quad v_r(\sigma,\theta) = 0$, (4)

far away from the bead: $\quad \lim_{r\to\infty} v_r = -u\cos\theta \ ; \quad \lim_{r\to\infty} v_\theta = u\sin\theta \quad \left(\lim_{r\to\infty}\vec{v} = \vec{u}\right).$ (5)

A sketch of $\vec{v}$:



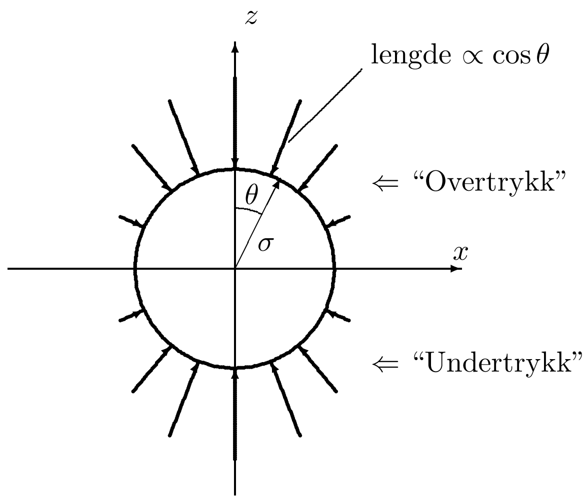
B) The pressure distribution on the bead surface is given by Eq. (5.50) with $r = \sigma$:

$$p(r=\sigma,\theta) = \left[\frac{3}{2}\eta_{\rm s}\, u\, \sigma^{-1} - \rho_{\rm m}\, g\, \sigma\right]\cos\theta + p_0, \tag{6}$$

where $\eta_{\rm s}$ is the fluid viscosity, g is the gravitational constant, $\rho_{\rm m}$ is the mass density of the fluid and p_0 is a constant. The term containing g represents the buoyancy. Relative to p_0, the pressure on the bead surface is

$$p(\sigma,\theta) - p_0 \quad \propto \quad \cos\theta, \tag{7}$$

represented by arrows in the sketch:



C) The geometry is the same when the solid sphere is replaced an air bubble, however, the boundary conditions are different. The requirement at $r \to \infty$ is as above, and the condition $v_r(\sigma) = 0$ is still valid as no fluid may cross into the bubble. However the non-slip condition $v_\theta(\sigma) = 0$ does not apply as there is no surface to "stick" on. In place, the shear force $\tau_{r\theta}$ along the surface must be zero at the bubble surface. The boundary conditions at the bead surface $r = \sigma$ therefore are

$$v_r(\sigma, \theta) = 0 \qquad \tau_{r\theta}(\sigma, \theta) = 0 \tag{8}$$

We may follow the solution in lecture notes up to where the boundary conditions are included. That is, we introduce the stream function Ψ by the definition

$$v_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \quad \text{and} \quad v_\theta = \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r} . \tag{9}$$

Ψ is assumed to be expressed

$$\Psi(r, \theta) = f(r) \cdot \sin^2\theta , \tag{10}$$

where the trial solution of $f(r)$ is

$$\begin{aligned} f(r) &= A_{-1}r^{-1} + A_1 r^1 + A_2 r^2 + A_4 r^4 \\ \Rightarrow f'(r) &= -A_{-1}r^{-2} + A_1 + 2A_2 r + 4A_4 r^3 \\ \Rightarrow f''(r) &= 2A_{-1}r^{-3} + 2A_2 + 12A_4 r^2 \end{aligned} \tag{11}$$

The boundary conditions for $r \to \infty$ at Eq. (5.42) yields $A_2 = \frac{u}{2}$ and $A_4 = 0$.

The boundary conditions at $r = \sigma$ at Eq. (8) inserted in Eq. (9) evaluate to:

$$v_r(\sigma, \theta) = 0 \quad \overset{\text{Eq. (9)}}{\Rightarrow} \quad \left.\frac{\partial \Psi}{\partial \theta}\right|_{r=\sigma} = 0 \quad \Rightarrow \quad 2\sin\theta\cos\theta f(\sigma) = 0 \quad \Rightarrow \quad f(\sigma) = 0 \tag{12}$$

$$\tau_{r\theta} = -\eta_s \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} \right]_{r=\sigma} = 0 \quad \Rightarrow \quad \sigma f'' - 2f' = 0 \quad \text{for} \quad r = \sigma . \tag{13}$$

Details of the last calculation (note that $v_r = 0$ for all θ):

$$\begin{aligned} \tau_{r\theta} &= -\eta_s \left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) \right]_{r=\sigma} = -\eta_s \left[r\frac{\partial}{\partial r}\left(\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial r}\right) \right]_{r=\sigma} \\ &= -\eta_s \left[r\frac{\partial}{\partial r}\left(\frac{f'(r)\sin\theta}{r^2}\right) \right]_{r=\sigma} = -\eta_s \left[r\frac{f''(r)}{r^2} - 2r\frac{f'(r)}{r^3} \right]_{r=\sigma} \sin\theta \\ &= -\eta_s \left[\frac{f''(\sigma)}{\sigma} - 2\frac{f'(\sigma)}{\sigma^2} \right] \sin\theta \end{aligned} \tag{14}$$

Eqs. (12) and (13) inserted in Eq. (11) make the coefficients A_{-1} and A_1 determined:

$$\begin{aligned} f(\sigma) &= A_{-1}\sigma^{-1} + A_1\sigma^1 + A_2\sigma^2 = 0 \\ \sigma f''(\sigma) - 2f'(\sigma) &= 2A_{-1}\sigma^{-2} + 2A_2\sigma + 2A_{-1}\sigma^{-2} - 2A_1 - 4A_2\sigma = 0 \\ &\Rightarrow A_{-1} = 0 \quad \wedge \quad A_1 = -A_2\sigma = -\frac{u}{2}\sigma \\ &\Rightarrow f(r) = \frac{u}{2}(r^2 - \sigma r) \quad \text{and} \quad \Psi(r, \theta) = \frac{u}{2}(r^2 - \sigma r)\sin^2\theta \end{aligned} \tag{15}$$

Inserted in Eq. (9)

$$v_r = -u\left[1-\frac{\sigma}{r}\right]\cos\theta \tag{16}$$

$$v_\theta = u\left[1-\frac{1}{2}\frac{\sigma}{r}\right]\sin\theta\,. \tag{17}$$

It is interesting to compare this velocity field to the velocity field around a rigid sphere in Eqs. (2) and (3).

To determine the friction coefficient, ζ, we need the total force, F_z, in z-direction, as $F_z = -\zeta\cdot u + F_{\text{buoyancy}}$. Integrate as for a solid bead (Eq. (5.51)):

$$F_z = \int_0^{2\pi}\int_0^{\pi}\left(-\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\pi}}_n\right)\Big|_{r=\sigma}\sigma^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi \tag{18}$$

where $\vec{\boldsymbol{\pi}}_n$ is the traction normal to the bubble and thus

$$\begin{aligned}-\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\pi}}_n = -\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\delta}}_r\cdot\vec{\vec{\boldsymbol{\pi}}} &= -\vec{\boldsymbol{\delta}}_z\cdot\left(\pi_{rr}\,\vec{\boldsymbol{\delta}}_r + \pi_{r\theta}\,\vec{\boldsymbol{\delta}}_\theta\right)\\ &= -(p+\tau_{rr})\,\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\delta}}_r - \tau_{r\theta}\,\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\delta}}_\theta\\ &= -(p+\tau_{rr})\,(-\cos\theta) - 0\cdot\,\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\delta}}_\theta\end{aligned} \tag{19}$$

where we have used $\vec{\boldsymbol{\delta}}_z = \vec{\boldsymbol{\delta}}_r\cos\theta - \vec{\boldsymbol{\delta}}_\theta\sin\theta$ and Eq. (8): $\tau_{r\theta}=0$.

The pressure $p(r,\theta)$ is determined from the equation of motion:

$$\begin{aligned}\frac{\partial p}{\partial r} &= \eta_{\mathrm{s}}\nabla^2 v_r + \rho_{\mathrm{m}}\vec{g}\cdot\vec{\delta}_r\\ &= \eta_{\mathrm{s}}\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)\right] - \rho_{\mathrm{m}} g\cos\theta\\ &\overset{(16)}{=} \eta_{\mathrm{s}}\left[\frac{1}{r^2}(-2u\cos\theta) + \frac{1}{r^2}2u\cdot\left(1-\frac{\sigma}{r}\right)\cos\theta\right] - \rho_{\mathrm{m}} g\cos\theta\\ &= -2u\eta_{\mathrm{s}}\frac{\sigma}{r^3}\cos\theta - \rho_{\mathrm{m}} g\cos\theta\,.\end{aligned} \tag{20}$$

Integration yields

$$p(r,\theta) = u\eta_{\mathrm{s}}\frac{\sigma}{r^2}\cos\theta - \rho_{\mathrm{m}} g r\cdot\cos\theta + p_0\,. \tag{21}$$

Finally we in Eq. (19) need τ_{rr}. From $\vec{\vec{\boldsymbol{\tau}}} = -\eta_{\mathrm{s}}\left(\nabla\vec{v} + (\nabla\vec{v})^{\mathrm{T}}\right)$ we obtain

$$\tau_{rr} = -2\eta_{\mathrm{s}}\frac{\partial v_r}{\partial r} = 2\eta_{\mathrm{s}} u\frac{\sigma}{r^2}\cos\theta\,. \tag{22}$$

Inserted in Eq. (19):

$$\begin{aligned}-\vec{\boldsymbol{\delta}}_z\cdot\vec{\boldsymbol{\pi}}_n &= -(u\eta_{\mathrm{s}}\frac{\sigma}{r^2}\cos\theta - \rho_{\mathrm{m}} g r\cdot\cos\theta + p_0 + 2\eta_{\mathrm{s}} u\frac{\sigma}{r^2}\cos\theta)\,(-\cos\theta)\\ &= 3u\eta_{\mathrm{s}}\frac{\sigma}{r^2}\cos^2\theta - \rho_{\mathrm{m}} g r\cdot\cos^2\theta - p_0\cos\theta\,,\end{aligned} \tag{23}$$

and the force in Eq. (18) is determined by integration, using

$$\int\!\!\int\cos\theta\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 0 \quad ; \quad \int\!\!\int\cos^2\theta\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 4\pi/3,$$

yielding at $r=\sigma$

$$F_z = -4\pi\eta_{\mathrm{s}}\sigma u + \frac{4\pi}{3}\sigma^3\rho_{\mathrm{m}}\,g\ . \tag{24}$$

The last term equals the buoyancy and the first the viscous drag where the (translational) friction coefficient is recognized as

$$\boxed{\zeta_{\text{bubble}} = 4\pi\eta_{\text{s}}\sigma} \tag{25}$$

Note that the friction of the air bubble is 4/6 of the friction of a solid bead of the same size.

The velocity of free rise is found by $F_z = 0$, yielding

$$u = \frac{\rho_{\text{m}}}{3\eta_{\text{s}}}\sigma^2 g.$$

At this value of u Eqs. (21) and (22) show that at the surface of the bubble the total traction normal to the bubble equals p_0 for all values of θ:

$$\begin{aligned} \pi_{rr} &= p + \tau_{rr} \\ &= \left(-\frac{2}{3}\rho_{\text{m}}\sigma g\cos\theta + p_0\right) + \frac{2}{3}\rho_{\text{m}}\sigma g\cos\theta \quad = \quad p_0\,. \end{aligned} \tag{26}$$

This means that the bubble will remain spherical. However, inertial forces (not included here) will tend to deform the bubble, and the bubble shape will be given as a balance among viscous, inertial, and surface tension forces.