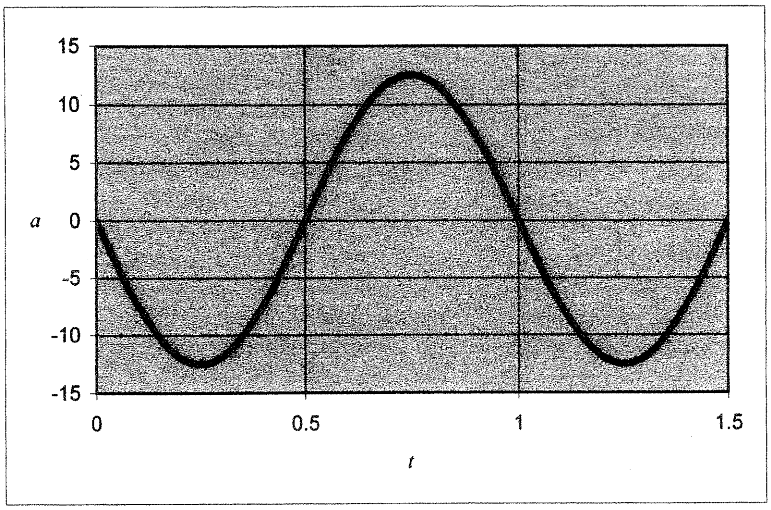

122 ••

Picture the Problem Because the acceleration of the rocket varies with time, it is not constant and integration of this function is required to determine the rocket's velocity and position as functions of time. The conditions on x and v at $t = 0$ are known as ***initial conditions.***

(*a*) Integrate $a(t)$ to find $v(t)$:

$$v(t) = \int a(t)\,dt = b\int t\,dt = \tfrac{1}{2}bt^2 + C$$

where C, the constant of integration, can be determined from the initial conditions.

Integrate $v(t)$ to find $x(t)$:

$$x(t) = \int v(t)dt = \int\left[\tfrac{1}{2}bt^2 + C\right]dt$$
$$= \tfrac{1}{6}bt^3 + Ct + D$$

where D is a second constant of integration.

Using the initial conditions, find the constants C and D:

$$v(0) = 0 \Rightarrow C = 0$$

and

$$x(0) = 0 \Rightarrow D = 0$$

$$\therefore \boxed{x(t) = \tfrac{1}{6}bt^3}$$

(*b*) Evaluate $v(5\text{ s})$ and $x(5\text{ s})$ with $C = D = 0$ and $b = 3\text{ m/s}^2$:

$$v(5\,\text{s}) = \frac{1}{2}(3\text{m/s}^2)(5\text{s})^2 = \boxed{37.5\text{ m/s}}$$

and

$$x(5\,\text{s}) = \frac{1}{6}(3\text{m/s}^2)(5\text{s})^3 = \boxed{62.5\text{ m}}$$

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Picture the Problem The acceleration is a function of time; therefore it is not constant. The instantaneous velocity can be determined by integration of the acceleration and the average velocity from the displacement of the particle during the given time interval.

(*a*) Because the acceleration is the derivative of the velocity, integrate the acceleration to find the ***instantaneous velocity*** $v(t)$.

$$a(t)=\frac{dv}{dt} \Rightarrow v(t)=\int_{v_0=0}^{v(t)} dv' = \int_{t_0=0}^{t} a(t')dt'$$

Calculate the instantaneous velocity using the acceleration given.

$$v(t)=\left(0.2\,\text{m/s}^3\right)\int_{t_0=0}^{t} t'\,dt'$$

and

$$\boxed{v(t)=\left(0.1\,\text{m/s}^3\right)t^2}$$

(*b*) To calculate the ***average velocity***, we need the displacement:

$$v(t)\equiv\frac{dx}{dt}\Rightarrow x(t)=\int_{x_0=0}^{x(t)} dx' = \int_{t_0=0}^{t} v(t')dt'$$

Because the velocity is the derivative of the displacement, integrate the velocity to find Δx.

$$x(t)=\left(0.1\text{m/s}^3\right)\int_{t_0=0}^{t} t'^2\,dt' = \left(0.1\text{m/s}^3\right)\frac{t^3}{3}$$

and

$$\begin{aligned}\Delta x &= x(7\text{ s})-x(2\text{ s})\\ &= \left(0.1\,\text{m/s}^3\right)\left[\frac{(7\text{ s})^3-(2\text{ s})^3}{3}\right]\\ &= 11.2\text{ m}\end{aligned}$$

Using the definition of the ***average velocity***, calculate v_{av}.

$$v_{\text{av}}=\frac{\Delta x}{\Delta t}=\frac{11.2\text{ m}}{5\text{ s}}=\boxed{2.23\text{ m/s}}$$

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Determine the Concept Because the acceleration is a function of time, it is not constant. Hence we'll need to integrate the acceleration function to find the velocity as a function of time and integrate the velocity function to find the position as a function of time. The important concepts here are the definitions of velocity, acceleration, and average velocity.

(*a*) Starting from $t_o = 0$, integrate the instantaneous acceleration to obtain the instantaneous velocity as a function of time:

From $a=\dfrac{dv}{dt}$

it follows that

$$\int_{v_0}^{v} dv' = \int_{0}^{t}(a_0+bt')dt'$$

and

$$\boxed{v=v_0+a_0t+\tfrac{1}{2}bt^2}$$

(*b*) Now integrate the instantaneous velocity to obtain the position as a function of time:

From $v = \dfrac{dx}{dt}$ it follows that

$$\int_{x_0}^{x} dx' = \int_{t_0=0}^{t} v(t')dt'$$

$$= \int_{t_0}^{t}\left(v_9 + a_0 t' + \frac{b}{2}t'^2\right)dt'$$

and

$$\boxed{x = x_0 + v_0 t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3}$$

(*c*) The definition of the average velocity is the ratio of the displacement to the total time elapsed:

$$v_{av} \equiv \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0} = \frac{v_0 t + \tfrac{1}{2}a_0 t^2 + \tfrac{1}{6}bt^3}{t}$$

and

$$\boxed{v_{av} = v_0 + \tfrac{1}{2}a_0 t + \tfrac{1}{6}bt^2}$$

Note that v_{av} is not the same as that due to constant acceleration:

$$\left(v_{\text{constant acceleration}}\right)_{av} = \frac{v_0 + v}{2}$$

$$= \frac{v_0 + \left(v_0 + a_0 t + \tfrac{1}{2}bt^2\right)}{2}$$

$$= v_0 + \tfrac{1}{2}a_0 t + \tfrac{1}{4}bt^2$$

$$\neq v_{av}$$

General Problems

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Picture the Problem The acceleration of the marble is constant. Because the motion is downward, choose a coordinate system with downward as the positive direction. The equation $g_{exp} = (1\text{ m})/(\Delta t)^2$ originates in the constant-acceleration equation $\Delta x = v_0\Delta t + \tfrac{1}{2}a(\Delta t)^2$. Because the motion starts from rest, the displacement of the marble is 1 m, the acceleration is the experimental value g_{exp}, and the equation simplifies to $g_{exp} = (1\text{ m})/(\Delta t)^2$.

Express the percent difference between the accepted and experimental values for the acceleration due to gravity:

$$\%\text{ difference} = \frac{\left|g_{accepted} - g_{exp}\right|}{g_{accepted}}$$

Using a constant-acceleration equation, express the velocity of the marble in terms of its initial velocity, acceleration, and displacement:

$$v_f^2 = v_0^2 + 2a\Delta y$$

or, because $v_0 = 0$ and $a = g$,

$$v_f^2 = 2g\Delta y$$

(*b*) Draw a tangent line at the origin and measure its rise and run. Use this ratio to obtain an approximate value for the slope at the origin:

The tangent line appears to, at least approximately, pass through the point (5, 4). Using the origin as the second point,

$$\Delta x = 4\text{ cm} - 0 = 4\text{ cm}$$

and

$$\Delta t = 5\text{ s} - 0 = 5\text{ s}$$

Therefore, the slope of the tangent line and the velocity of the body as it passes through the origin is approximately:

$$v(0) = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{4\text{ cm}}{5\text{ s}} = \boxed{0.800\text{ cm/s}}$$

(*c*) Calculate the average velocity for the series of time intervals given by completing the table shown below:

t_0 (s)	t (s)	Δt (s)	x_0 (cm)	x (cm)	Δx (cm)	$v_{av}=\Delta x/\Delta t$ (m/s)
0	6	6	0	4.34	4.34	0.723
0	3	3	0	2.51	2.51	0.835
0	2	2	0	1.71	1.71	0.857
0	1	1	0	0.871	0.871	0.871
0	0.5	0.5	0	0.437	0.437	0.874
0	0.25	0.25	0	0.219	0.219	0.875

(*d*) Express the time derivative of the position:

$$\frac{dx}{dt} = A\omega\cos\omega t$$

Substitute numerical values and evaluate $\frac{dx}{dt}$ at $t = 0$:

$$\frac{dx}{dt} = A\omega\cos 0 = A\omega$$

$$= (0.05\,\text{m})(0.175\,\text{s}^{-1})$$

$$= \boxed{0.875\,\text{cm/s}}$$

(*e*) Compare the average velocities from part (*c*) with the instantaneous velocity from part (*d*):

As Δt, and thus Δx, becomes small, the value for the average velocity approaches that for the instantaneous velocity obtained in part (*d*). For $\Delta t = 0.25$ s, they agree to three significant figures.

127 •••

Determine the Concept Because the velocity varies nonlinearly with time, the acceleration of the object is not constant. We can find the acceleration of the object by differentiating its velocity with respect to time and its position function by integrating the velocity function. The important concepts here are the definitions of acceleration and velocity.

(*a*) The acceleration of the object is the derivative of its velocity with respect to time:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[v_{max}\sin(\omega t)\right]$$

$$= \boxed{\omega\, v_{max}\cos(\omega t)}$$

Because *a* varies sinusoidally with time it is *not* constant.

(*b*) Integrate the velocity with respect to time from 0 to *t* to obtain the change in position of the body:

$$\int_{x_0}^{x} dx' = \int_{t_0}^{t}\left[v_{max}\sin(\omega t')\right]dt'$$

and

$$x - x_0 = \left[\frac{-v_{max}}{\omega}\cos(\omega t')\right]_0^t$$

$$= \frac{-v_{max}}{\omega}\cos(\omega t) + \frac{v_{max}}{\omega}$$

or

$$\boxed{x = x_0 + \frac{v_{max}}{\omega}\left[1 - \cos(\omega t)\right]}$$

Note that, as given in the problem statement, $x(0\text{ s}) = x_0$.

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Picture the Problem Because the acceleration of the particle is a function of its position, it is not constant. Changing the variable of integration in the definition of acceleration will allow us to determine its velocity and position as functions of position.

(*a*) Because $a = dv/dt$, we must integrate to find $v(t)$. Because *a* is given as a function of *x*, we'll need to change variables in order to carry out the integration. Once we've changed variables, we'll separate them with *v* on the left side of the equation and *x* on the right:

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \left(2\,\text{s}^{-2}\right)x$$

or, upon separating variables,

$$vdv = \left(2\,\text{s}^{-2}\right)xdx$$

Integrate from x_o and v_o to *x* and *v*:

$$\int_{v_0=0}^{v} v'dv' = \int_{x_0}^{x}\left(2\,\text{s}^{-2}\right)x'dx'$$

and

$$v^2 - v_0^2 = \left(2\,\text{s}^{-2}\right)\left(x^2 - x_0^2\right)$$

Solve for *v* to obtain:

$$v = \sqrt{v_0^2 + \left(2\,\text{s}^{-2}\right)\left(x^2 - x_0^2\right)}$$

We are given that $x = bv$, where $b = 1$ s. Substitute for v and separate variables to obtain:

$$\frac{dx}{dt} = \frac{x}{b} \Rightarrow dt = b\frac{dx}{x}$$

Integrate and solve for $x(t)$:

$$\int_{t_0}^{t} dt' = b\int_{x_0}^{x} \frac{dx'}{x'} \Rightarrow (t - t_0) = b\ln\left(\frac{x}{x_0}\right)$$

and

$$x(t) = \boxed{x_0 e^{(t-t_0)/b}}$$

(*b*) Differentiate twice to obtain $v(t)$ and $a(t)$:

$$v = \frac{dx}{dt} = \frac{1}{b}x_0 e^{(t-t_0)/b}$$

and

$$a = \frac{dv}{dt} = \frac{1}{b^2}x_0 e^{(t-t_0)/b}$$

Substitute the result in part (*a*) to obtain the desired results:

$$v(t) = \frac{1}{b}x(t)$$

and

$$a(t) = \frac{1}{b^2}x(t)$$

so

$$a(t) = \frac{1}{b}v(t) = \frac{1}{b^2}x(t)$$

Because the numerical value of b, expressed in SI units, is one, the numerical values of a, v, and x are the same at each instant in time.

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Picture the Problem Because the acceleration of the rock is a function of time, it is not constant. Choose a coordinate system in which downward is positive and the origin at the point of release of the rock.

Separate variables in $a(t) = dv/dt = ge^{-bt}$ to obtain:

$$dv = ge^{-bt}dt$$

Integrate from $t_o = 0$, $v_o = 0$ to some later time t and velocity v:

$$v = \int_0^v dv' = \int_0^t ge^{-bt'}dt' = \frac{g}{-b}\left[e^{-bt'}\right]_0^t$$

$$= \frac{g}{b}\left(1 - e^{-bt}\right) = v_{\text{term}}\left(1 - e^{-bt}\right)$$

where

$$v_{\text{term}} = \frac{g}{b}$$

Separate variables in $v = dy/dt = v_{\text{term}}\left(1-e^{-bt}\right)$ to obtain:

$$dy = v_{\text{term}}\left(1-e^{-bt}\right)dt$$

Integrate from $t_0 = 0$, $y_0 = 0$ to some later time t and position y:

$$\int_0^y dy' = \int_0^t v_{\text{term}}\left(1-e^{-bt'}\right)dt'$$

$$y = v_{\text{term}}\left[t' + \frac{1}{b}e^{-bt'}\right]_0^t$$

$$= \boxed{v_{\text{term}}t - \frac{v_{\text{term}}}{b}\left(1-e^{-bt}\right)}$$

This last result is very interesting. It says that throughout its free-fall, the object experiences drag; therefore it has not fallen as far at any given time as it would have if it were falling at the constant velocity, v_{term}.

On the other hand, just as the velocity of the object asymptotically approaches v_{term}, the distance it has covered during its free-fall as a function of time asymptotically approaches the distance it would have fallen if it had fallen with v_{term} throughout its motion.

$$y\left(t_{\text{large}}\right) \to v_{\text{term}}t - \frac{v}{b} \to v_{\text{term}}t$$

This should not be surprising because in the expression above, the first term grows linearly with time while the second term approaches a constant and therefore becomes less important with time.

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Picture the Problem Because the acceleration of the rock is a function of its velocity, it is not constant. Choose a coordinate system in which downward is positive and the origin is at the point of release of the rock.

Rewrite $a = g - bv$ explicitly as a differential equation:

$$\frac{dv}{dt} = g - bv$$

Separate the variables, v on the left, t on the right:

$$\frac{dv}{g - bv} = dt$$

Integrate the left-hand side of this equation from 0 to v and the right-hand side from 0 to t:

$$\int_0^v \frac{dv'}{g - bv'} = \int_0^t dt'$$

and

$$-\frac{1}{b}\ln\left(\frac{g - bv}{g}\right) = t$$

Solve this expression for v.

$$v = \frac{g}{b}\left(1 - e^{-bt}\right)$$

Finally, differentiate this expression with respect to time to obtain an expression for the acceleration and complete the proof.

$$a = \frac{dv}{dt} = \boxed{ge^{-bt}}$$

132 •••

Picture the Problem The skydiver's acceleration is a function of her velocity; therefore it is not constant. Expressing her acceleration as the derivative of her velocity, separating the variables, and then integrating will give her velocity as a function of time.

(*a*) Rewrite $a = g - cv^2$ explicitly as a differential equation:

$$\frac{dv}{dt} = g - cv^2$$

Separate the variables, with v on the left, and t on the right:

$$\frac{dv}{g - cv^2} = dt$$

Eliminate c by using $c = \frac{g}{v_T^2}$:

$$\frac{dv}{g - \frac{g}{v_T^2}v^2} = \frac{dv}{g\left[1 - \left(\frac{v}{v_T}\right)^2\right]} = dt$$

or

$$\frac{dv}{1 - \left(\frac{v}{v_T}\right)^2} = gdt$$

Integrate the left-hand side of this equation from 0 to v and the right-hand side from 0 to t:

$$\int_0^v \frac{dv'}{1 - \left(\frac{v'}{v_T}\right)^2} = g\int_0^t dt' = gt$$

The integral can be found in integral tables:

$$v_T \tanh^{-1}(v/v_T) = gt$$

or

$$\tanh^{-1}(v/v_T) = (g/v_T)t$$