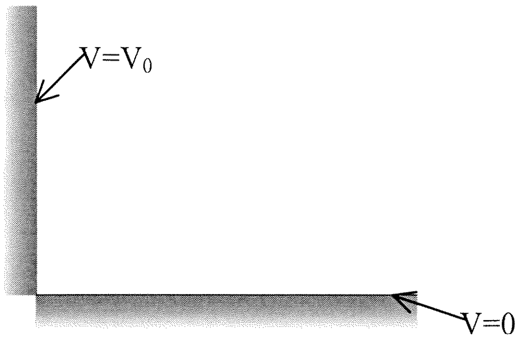
TFY 4240

Øving 3 2006, Solution

Problem 1



a) We start by noting that

$$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}\frac{du}{dx}$$

Thus

$$\frac{\partial V}{\partial x} = -\frac{2V_0}{\pi}\frac{y}{x^2+y^2}; \qquad \frac{\partial V}{\partial y} = \frac{2V_0}{\pi}\frac{x}{x^2+y^2}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{2V_0}{\pi}\frac{2xy}{(x^2+y^2)^2}; \qquad \frac{\partial^2 V}{\partial y^2} = -\frac{2V_0}{\pi}\frac{2xy}{(x^2+y^2)^2}$$

giving

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Now look at the boundary conditions:

X=0

$$V(0,y) = \frac{2V_0}{\pi}Arctg(\infty) = V_0$$

$$V(x,0) = \frac{2V_0}{\pi}Arctg(0) = 0$$

The induced charge on the vertical plate is given by

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0} = \frac{2V_0}{\pi}\frac{\varepsilon_0}{y}$$

and for the horisontal plate

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial y}\Big|_{y=0} = -\frac{2V_0}{\pi}\frac{\varepsilon_0}{x}$$

Problem 3.12

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.30)}, \quad \text{where} \quad C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy \quad \text{(Eq. 3.34)}.$$

In this case $V_0(y) = \left\{ \begin{array}{ll} +V_0, & \text{for } 0 < y < a/2 \\ -V_0, & \text{for } a/2 < y < a \end{array} \right\}$. Therefore,

$$\begin{aligned} C_n &= \frac{2}{a}V_0\left\{\int_0^{a/2}\sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right\} = \frac{2V_0}{a}\left\{-\left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_0^{a/2} + \left.\frac{\cos(n\pi y/a)}{(n\pi/a)}\right|_{a/2}^{a}\right\} \\ &= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right) + \cos(0) + \cos(n\pi) - \cos\left(\frac{n\pi}{2}\right)\right\} = \frac{2V_0}{n\pi}\left\{1 + (-1)^n - 2\cos\left(\frac{n\pi}{2}\right)\right\}. \end{aligned}$$

The term in curly brackets is:

$$\left\{ \begin{array}{lll} n=1 & : & 1-1-2\cos(\pi/2) = 0, \\ n=2 & : & 1+1-2\cos(\pi) = 4, \\ n=3 & : & 1-1-2\cos(3\pi/2) = 0, \\ n=4 & : & 1+1-2\cos(2\pi) = 0, \end{array} \right\} \text{etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

Therefore

$$C_n = \left\{ \begin{array}{ll} 8V_0/n\pi, & n = 2, 6, 10, 14, \text{etc. (in general, } 4j+2, \text{ for } j = 0, 1, 2, \ldots), \\ 0, & \text{otherwise.} \end{array} \right.$$

So

$$\boxed{V(x,y) = \frac{8V_0}{\pi}\sum_{n=2,6,10,\ldots}\frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}} = \frac{8V_0}{\pi}\sum_{j=0}^{\infty}\frac{e^{-(4j+2)\pi x/a}\sin[(4j+2)\pi y/a]}{(4j+2)}.$$

Problem 3.32

$Q = -q$, so $V_{\text{mono}} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r}$; $\mathbf{p} = qa\,\hat{\mathbf{z}}$, so $V_{\text{dip}} = \frac{1}{4\pi\epsilon_0}\frac{qa\cos\theta}{r^2}$. Therefore

$$\boxed{V(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r} + \frac{a\cos\theta}{r^2}\right).} \quad \boxed{\mathbf{E}(r,\theta) \cong \frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r^2}\,\hat{\mathbf{r}} + \frac{a}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right].}$$

Problem 3.33

$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 3.36). So $3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}} = 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$. So Eq. 3.104 ≡ Eq. 3.103. ✓