TFY 4240
Øving 4 2006, Solution

Problem 1

a)
The potential outside is given by:

$$V(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

and inside

$$V(r,\theta) = \sum A_l r^l P_l(\cos\theta)$$

$$V_{inside} = V_{outside}\big|_{r=R} \quad gives \quad A_l = \frac{B_l}{R^{2l+1}}$$

At the surface

$$V(R,\theta) = \sum \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0 \cos^2\theta$$

with

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V(R,\theta) P_l(\cos\theta) \sin\theta d\theta$$

We can write the potential at the surface of the sphere as

$$V(R,\theta) = V_0 \cos^2\theta = V_0\left(\frac{3\cos^2\theta - 1}{3} + \frac{1}{3}\right) = V_0\left(\frac{2}{3} P_2(\cos\theta) + \frac{1}{3} P_0(\cos\theta)\right)$$

which gives

$$B_l = \frac{2l+1}{2} R^{l+1} V_0 \int_0^{\pi} \left(\frac{2}{3} P_2(\cos\theta) + \frac{1}{3} P_0(\cos\theta)\right) P_l(\cos\theta) \sin\theta d\theta$$

giving

$$B_0 = \frac{1}{3} R V_0$$

$$B_2 = \frac{2}{3} R^3 V_0$$

Therefore:

$$V_{outside} = V_0\left(\frac{1}{3}\frac{R}{r} P_0(\cos\theta) + \frac{2}{3}\frac{R^3}{r^3} P_2(\cos\theta)\right)$$

b) The surface charge is determined by the equation:

$$\sigma(\theta) = -\varepsilon_0 (\frac{\partial V_{outside}}{\partial r} - \frac{\partial V_{inside}}{\partial r})$$

We had

$$outside$$

$$V(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$and \ \ inside$$

$$V(r,\theta) = \sum A_l r^l P_l(\cos\theta) = \sum \frac{B_l r^l}{R^{2l+1}} P_l(\cos\theta)$$

Inserting this in the expression for σ

$$\sigma(\theta) = -\varepsilon_0 (\sum -(l+1) \frac{B_l}{r^{l+2}} P_l(\cos\theta) - l \frac{B_l r^{l-1}}{R^{2l+1}} P_l(\cos\theta))\Big|_{r=R}$$

$$= \varepsilon_0 \sum (2l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta)$$

Inserting for B_l:

$$\sigma(\theta) = \varepsilon_0 (\frac{B_0 P_0(\cos\theta)}{R^2} + \frac{B_2 P_2(\cos\theta)}{R^4})$$

$$= \frac{\varepsilon_0 V_0}{R} (\frac{1}{3} P_0(\cos\theta) + \frac{10}{3} P_2(\cos\theta))$$

$$= \frac{\varepsilon_0 V_0}{3R} (15\cos^2\theta - 4)$$

c) The quadrupole moment is given by:

$$p = \int r^2 P_2(\cos\theta) dq = 2\pi R^4 \int P_2(\cos\theta) \sigma(\theta) \sin\theta d\theta$$

$$= 2\pi R^4 \int_0^\pi P_2(\cos\theta) \frac{\varepsilon_0 V_0}{R} (\frac{1}{3} P_0(\cos\theta) + \frac{10}{3} P_2(\cos\theta)) \sin\theta d\theta$$

$$= 4\pi\varepsilon_0 \frac{2}{3} V_0 R^3$$

This is, apart from the $4\pi\varepsilon_0$ term, the same as the coefficient of $P_2(\cos\theta)$ in the expression for the potential, which is given by:

$$V_{quadru}(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{p_2}{r^3} P_2(\cos\theta)$$

Problem 4.14

Total charge on the dielectric is $Q_{\text{tot}} = \oint_S \sigma_b \, da + \int_\mathcal{V} \rho_b \, d\tau = \oint_S \mathbf{P} \cdot d\mathbf{a} - \int_\mathcal{V} \nabla\cdot\mathbf{P} \, d\tau$. But the divergence theorem says $\oint_S \mathbf{P} \cdot d\mathbf{a} = \int_\mathcal{V} \nabla\cdot\mathbf{P} \, d\tau$, so $Q_{\text{enc}} = 0$. qed

Problem 4.21

Let Q be the charge on a length ℓ of the inner conductor.

$$\oint \mathbf{D} \cdot d\mathbf{a} = D 2\pi s\ell = Q \Rightarrow D = \frac{Q}{2\pi s\ell}; \quad E = \frac{Q}{2\pi\epsilon_0 s\ell} \ (a < s < b), \quad E = \frac{Q}{2\pi\epsilon s\ell} \ (b < r < c).$$

$$V = -\int_c^a \mathbf{E} \cdot d\mathbf{l} = \int_a^b \left(\frac{Q}{2\pi\epsilon_0 \ell}\right) \frac{ds}{s} + \int_b^c \left(\frac{Q}{2\pi\epsilon\ell}\right) \frac{ds}{s} = \frac{Q}{2\pi\epsilon_0 \ell}\left[\ln\left(\frac{b}{a}\right) + \frac{\epsilon_0}{\epsilon} \ln\left(\frac{c}{b}\right)\right].$$

$$\frac{C}{\ell} = \frac{Q}{V\ell} = \boxed{\frac{2\pi\epsilon_0}{\ln(b/a) + (1/\epsilon_r)\ln(c/b)}}.$$

Problem 4.30

Net force is $\boxed{\text{to the right}}$ (see diagram). Note that the field lines must bulge to the right, as shown, because E is perpendicular to the surface of each conductor.