TFY 4240 Løsning Øving 9

Problem 1

Problem 9.19

(a) Use the binomial expansion for the square root in Eq. 9.126:

$$\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2-1\right]^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}.$$

So (Eq. 9.128) $d = \frac{1}{\kappa} \cong \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$. qed

For pure water, $\begin{cases} \epsilon = \epsilon_r\epsilon_0 = 80.1\,\epsilon_0 & \text{(Table 4.2)}, \\ \mu = \mu_0(1+\chi_m) = \mu_0(1-9.0\times10^{-6}) \cong \mu_0 & \text{(Table 6.1)}, \\ \sigma = 1/(2.5\times10^5) & \text{(Table 7.1)}. \end{cases}$

So $d = (2)(2.5\times10^5)\sqrt{\frac{(80.1)(8.85\times10^{-12})}{4\pi\times10^{-7}}} = \boxed{1.19\times10^4\text{ m.}}$

(b) In this case $(\sigma/\epsilon\omega)^2$ dominates, so (Eq. 9.126) $k \cong \kappa$, and hence (Eqs. 9.128 and 9.129) $\lambda = \frac{2\pi}{k} \cong \frac{2\pi}{\kappa} = 2\pi d$, or $d = \frac{\lambda}{2\pi}$. qed

Meanwhile $\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{(10^{15})(4\pi\times10^{-7})(10^7)}{2}} = 8\times10^7$; $d = \frac{1}{\kappa} = \frac{1}{8\times10^7} = 1.3\times10^{-8} = \boxed{13\text{ nm.}}$ So the fields do not penetrate far into a metal—which is what accounts for their opacity.

(c) Since $k \cong \kappa$, as we found in (b), Eq. 9.134 says $\phi = \tan^{-1}(1) = 45°$. qed

Meanwhile, Eq. 9.137 says $\frac{B_0}{E_0} \cong \sqrt{\epsilon\mu\frac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\frac{\sigma\mu}{\omega}}}$. For a typical metal, then, $\frac{B_0}{E_0} = \sqrt{\frac{(10^7)(4\pi\times10^{-7})}{10^{15}}} = \boxed{10^{-7}\text{ s/m.}}$ (In vacuum, the ratio is $1/c = 1/(3\times10^8) = 3\times10^{-9}$ s/m, so the magnetic field is comparatively about 100 times larger in a metal.)

Problem 2

a) The transfer matrix is simple to use to see what happens in the limit $d \to 0$. Assume that we have several films, each of thickness $d \ll \lambda$ for each film. We expand the sine and cosine terms in the transfer matrix, giving

$$\overline{\overline{M}}_i \approx \begin{bmatrix} 1 & \frac{-i}{n_i}k_i d_i \\ -in_i k_i d_i & 1 \end{bmatrix}$$

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$$\overline{\overline{M}} = \overline{\overline{M}}_1\overline{\overline{M}}_2\cdots \approx \begin{bmatrix} 1 & \sum_i^N \frac{-i}{n_i}k_i d_i \\ -\sum_i^N i n_i k_i d_i & 1 \end{bmatrix}$$

to first order in $k_i d_i$.

Now $k_i = n_i k_o$, thus

$$\overline{\overline{M}} \approx \begin{bmatrix} 1 & -ik_0 \sum_i^N d_i \\ -ik_o \sum_i^N \varepsilon_i d_i & 1 \end{bmatrix} \qquad \varepsilon_i = n_i^2$$

Inserted in the formula for the reflection coefficient

$$A = D = 1, \quad B = -ik_o \sum d_i, \quad C = -ik_o \sum \varepsilon_i d_i$$

$$r = \frac{1 + Bn_T - C - n_T}{1 + Bn_T + C + n_T}$$

B and C are small and we can expand further: ($\varepsilon_T = n_T^2$)

$$r = \frac{1 - n_T}{1 + n_T}\left[1 + 2ik_0 \sum_i d_i \frac{\varepsilon_i - \varepsilon_T}{1 - \varepsilon_T}\right]$$

$$r = r_0\left[1 + 2ik_0 \sum_i d_i \frac{\varepsilon_i - \varepsilon_T}{1 - \varepsilon_T}\right]$$

You get the same result if you start with the equation

$$r = \frac{r_{01} + r_{12}e^{2i\delta_1}}{1 + r_{12}r_{01}e^{2i\delta_1}}$$

and expand the exponentials.

$$r = r_0\left[1 + d\,\frac{\varepsilon_i - \varepsilon_T}{1 - \varepsilon_T}\right] = r_0\left[1 + 2i\frac{2\pi}{500 \cdot 10^{-9}} 10 \cdot 10^{-9} \frac{3 - 37 - i9.2}{1 - 37 - i9.2}\right]$$
$$\approx r_0\left[0.99 + i0.22\right]$$

$$R = |r|^2 = R_0 \cdot 1.028$$

Thus, the film leads to a 2.8% increase in reflectivity.

Problem 3

The total transfer matrix will be

$$\overline{\overline{M}} = \begin{bmatrix} 0 & \frac{-i}{n_1} \\ -in_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{-i}{n_2} \\ -in_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{-i}{n_1} \\ -in_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-in_2}{n_1^2} \\ -i\frac{n_1^2}{n_2} & 0 \end{bmatrix}$$

and the reflectivity

$$r = \frac{n_T n_0 \frac{n_2}{n_1^2} - \frac{n_1^2}{n_2}}{n_T n_0 \frac{n_2}{n_1^2} + \frac{n_1^2}{n_2}}$$

which is zero for

$$n_T n_0 n_2^2 = n_1^4$$

For the given numbers we obtain a reflectivity R

R=0.004