# TFY 4240 Løsning Øving 10

## Problem 1

**Problem 9.37**

(a) Equation 9.91 $\Rightarrow \tilde{\mathbf{E}}_T(\mathbf{r},t) = \tilde{\mathbf{E}}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}$; $\mathbf{k}_T\cdot\mathbf{r} = k_T(\sin\theta_T\,\hat{\mathbf{x}} + \cos\theta_T\,\hat{\mathbf{z}})\cdot(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = k_T(x\sin\theta_T + z\cos\theta_T) = xk_T\sin\theta_T + izk_T\sqrt{\sin^2\theta_T - 1} = kx + i\kappa z$, where

$$\begin{aligned} k &\equiv k_T\sin\theta_T = \left(\frac{\omega n_2}{c}\right)\frac{n_1}{n_2}\sin\theta_I = \frac{\omega n_1}{c}\sin\theta_I, \\ \kappa &\equiv k_T\sqrt{\sin^2\theta_T - 1} = \frac{\omega n_2}{c}\sqrt{(n_1/n_2)^2\sin^2\theta_I - 1} = \frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_I - n_2^2}. \quad \text{So} \\ \tilde{\mathbf{E}}_T(\mathbf{r},t) &= \tilde{\mathbf{E}}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}. \quad \text{qed} \end{aligned}$$

(b) $R = \left|\frac{\tilde{E}_{0_R}}{\tilde{E}_{0_I}}\right|^2 = \left|\frac{\alpha-\beta}{\alpha+\beta}\right|^2$. Here $\beta$ is real (Eq. 9.106) and $\alpha$ is purely imaginary (Eq. 9.108); write $\alpha = ia$, with $a$ real: $R = \left(\frac{ia-\beta}{ia+\beta}\right)\left(\frac{-ia-\beta}{-ia+\beta}\right) = \frac{a^2+\beta^2}{a^2+\beta^2} = \boxed{1.}$

(c) From Prob. 9.16, $E_{0_R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0_I}$, so $R = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|^2 = \left|\frac{1-ia\beta}{1+ia\beta}\right|^2 = \frac{(1-ia\beta)(1+ia\beta)}{(1+ia\beta)(1-ia\beta)} = \boxed{1.}$

(d) From the solution to Prob. 9.16, the transmitted wave is

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}(-\cos\theta_T\,\hat{\mathbf{x}} + \sin\theta_T\,\hat{\mathbf{z}}).$$

Using the results in (a): $\mathbf{k}_T\cdot\mathbf{r} = kx + i\kappa z - \omega t$, $\sin\theta_T = \frac{ck}{\omega n_2}$, $\cos\theta_T = i\frac{c\kappa}{\omega n_2}$:

$$\tilde{\mathbf{E}}(\mathbf{r},t) = \tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\,\hat{\mathbf{y}}, \quad \tilde{\mathbf{B}}(\mathbf{r},t) = \frac{1}{v_2}\tilde{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)}\left(-i\frac{c\kappa}{\omega n_2}\,\hat{\mathbf{x}} + \frac{ck}{\omega n_2}\,\hat{\mathbf{z}}\right).$$

We may as well choose the phase constant so that $\tilde{E}_{0_T}$ is *real*. Then

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{y}}; \\ \mathbf{B}(\mathbf{r},t) &= \frac{1}{v_2}E_0 e^{-\kappa z}\frac{c}{\omega n_2}\mathrm{Re}\left\{[\cos(kx-\omega t) + i\sin(kx-\omega t)]\,[-i\kappa\,\hat{\mathbf{x}} + k\,\hat{\mathbf{z}}]\right\} \\ &= \frac{1}{\omega}E_0 e^{-\kappa z}\left[\kappa\sin(kx-\omega t)\,\hat{\mathbf{x}} + k\cos(kx-\omega t)\,\hat{\mathbf{z}}\right]. \quad \text{qed} \end{aligned}$$

(I used $v_2 = c/n_2$ to simplfy $\mathbf{B}$.)

(e)

$$\begin{aligned} \text{(i)}\ \nabla\cdot\mathbf{E} &= \frac{\partial}{\partial y}\left[E_0 e^{-\kappa z}\cos(kx-\omega t)\right] = 0. \checkmark \\ \text{(ii)}\ \nabla\cdot\mathbf{B} &= \frac{\partial}{\partial x}\left[\frac{E_0}{\omega}e^{-\kappa z}\kappa\sin(kx-\omega t)\right] + \frac{\partial}{\partial z}\left[\frac{E_0}{\omega}e^{-\kappa z}k\cos(kx-\omega t)\right] \\ &= \frac{E_0}{\omega}\left[e^{-\kappa z}\kappa k\cos(kx-\omega t) - \kappa e^{-\kappa z}k\cos(kx-\omega t)\right] = 0. \checkmark \\ \text{(iii)}\ \nabla\times\mathbf{E} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & E_y & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}\,\hat{\mathbf{x}} + \frac{\partial E_y}{\partial x}\,\hat{\mathbf{z}} \\ &= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - E_0 e^{-\kappa z}k\sin(kx-\omega t)\,\hat{\mathbf{z}}. \\ -\frac{\partial\mathbf{B}}{\partial t} &= -\frac{E_0}{\omega}e^{-\kappa z}\left[-\kappa\omega\cos(kx-\omega t)\,\hat{\mathbf{x}} + k\omega\sin(kx-\omega t)\,\hat{\mathbf{z}}\right] \\ &= \kappa E_0 e^{-\kappa z}\cos(kx-\omega t)\,\hat{\mathbf{x}} - kE_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{z}} = \nabla\times\mathbf{E}. \checkmark \\ \text{(iv)}\ \nabla\times\mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & 0 & B_z \end{vmatrix} = \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\hat{\mathbf{y}} \\ &= \left[-\frac{E_0}{\omega}\kappa^2 e^{-\kappa z}\sin(kx-\omega t) + \frac{E_0}{\omega}e^{-\kappa z}k^2\sin(kx-\omega t)\right]\hat{\mathbf{y}} = (k^2-\kappa^2)\frac{E_0}{\omega}e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}. \end{aligned}$$

$$\text{Eq. 9.202} \Rightarrow k^2 - \kappa^2 = \left(\frac{\omega}{c}\right)^2\left[n_1^2\sin^2\theta_I - (n_1\sin\theta_I)^2 + (n_2)^2\right] = \left(\frac{n_2\omega}{c}\right)^2 = \omega^2\epsilon_2\mu_2.$$

$$= \epsilon_2\mu_2\omega E_0 e^{-\kappa z}\sin(kx-\omega t)\,\hat{\mathbf{y}}.$$

$$\mu_2\epsilon_2\frac{\partial \mathbf{E}}{\partial t} = \mu_2\epsilon_2 E_0 e^{-\kappa z}\omega\sin(kx-\omega t)\,\hat{\mathbf{y}} = \nabla\times\mathbf{B}\ \checkmark.$$

(f)

$$\mathbf{S} = \frac{1}{\mu_2}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_2}\frac{E_0^2}{\omega}e^{-2\kappa z}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ 0 & \cos(kx-\omega t) & 0\\ \kappa\sin(kx-\omega t) & 0 & k\cos(kx-\omega t)\end{vmatrix}$$

$$= \boxed{\frac{E_0^2}{\mu_2\omega}e^{-2\kappa z}\left[k\cos^2(kx-\omega t)\,\hat{\mathbf{x}} - \kappa\sin(kx-\omega t)\cos(kx-\omega t)\,\hat{\mathbf{z}}\right].}$$

Averaging over a complete cycle, using $\langle\cos^2\rangle = 1/2$ and $\langle\sin\cos\rangle = 0$, $\langle\mathbf{S}\rangle = \dfrac{E_0^2 k}{2\mu_2\omega}e^{-2\kappa z}\,\hat{\mathbf{x}}$. On average, then, no energy is transmitted in the $z$ direction, only in the $x$ direction (parallel to the interface). qed

## Problem 9.38

Look for solutions of the form $\mathbf{E} = \mathbf{E}_0(x,y,z)e^{-i\omega t}$, $\mathbf{B} = \mathbf{B}_0(x,y,z)e^{-i\omega t}$, subject to the boundary conditions $\mathbf{E}^{\parallel} = 0$, $B^{\perp} = 0$ at all surfaces. Maxwell's equations, in the form of Eq. 9.177, give

$$\left\{\begin{array}{llll}\nabla\cdot\mathbf{E} = 0 & \Rightarrow \nabla\cdot\mathbf{E}_0 = 0; & \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & \Rightarrow \nabla\times\mathbf{E}_0 = i\omega\mathbf{B}_0;\\ \nabla\cdot\mathbf{B} = 0 & \Rightarrow \nabla\cdot\mathbf{B}_0 = 0; & \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} & \Rightarrow \nabla\times\mathbf{B}_0 = -\frac{i\omega}{c^2}\mathbf{E}_0.\end{array}\right\}$$

From now on I'll leave off the subscript (0). The problem is to solve the (time independent) equations

$$\left\{\begin{array}{ll}\nabla\cdot\mathbf{E} = 0; & \nabla\times\mathbf{E} = i\omega\mathbf{B};\\ \nabla\cdot\mathbf{B} = 0; & \nabla\times\mathbf{B} = -\frac{i\omega}{c^2}\mathbf{E}.\end{array}\right\}$$

From $\nabla\times\mathbf{E} = i\omega\mathbf{B}$ it follows that I can get $\mathbf{B}$ once I know $\mathbf{E}$, so I'll concentrate on the latter for the moment.

$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = \nabla\times(i\omega\mathbf{B}) = i\omega\left(-\frac{i\omega}{c^2}\mathbf{E}\right) = \frac{\omega^2}{c^2}\mathbf{E}$. So $\nabla^2 E_x = -\left(\frac{\omega}{c}\right)^2 E_x$; $\nabla^2 E_y = -\left(\frac{\omega}{c}\right)^2 E_y$; $\nabla^2 E_z = -\left(\frac{\omega}{c}\right)^2 E_z$. Solve each of these by separation of variables: $E_x(x,y,z) = X(x)Y(y)Z(z) \Rightarrow YZ\frac{d^2X}{dx^2} + ZX\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2} = -\left(\frac{\omega}{c}\right)^2 XYZ$, or $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -(\omega/c)^2$. Each term must be a constant, so $\frac{d^2X}{dx^2} = -k_x^2 X$, $\frac{d^2Y}{dy^2} = -k_y^2 Y$, $\frac{d^2Z}{dz^2} = -k_z^2 Z$, with $k_x^2 + k_y^2 + k_z^2 = -(\omega/c)^2$. The solution is

$$E_x(x,y,z) = [A\sin(k_x x) + B\cos(k_x x)][C\sin(k_y y) + D\cos(k_y y)][E\sin(k_z z) + F\cos(k_z z)].$$

But $\mathbf{E}^{\parallel} = 0$ at the boundaries $\Rightarrow E_x = 0$ at $y = 0$ and $z = 0$, so $D = F = 0$, and $E_x = 0$ at $y = b$ and $z = d$, so $k_y = n\pi/b$ and $k_z = l\pi/d$, where $n$ and $l$ are integers. A similar argument applies to $E_y$ and $E_z$. *Conclusion:*

$$\begin{aligned} E_x(x,y,z) &= [A\sin(k_x x) + B\cos(k_x x)]\sin(k_y y)\sin(k_z z),\\ E_y(x,y,z) &= \sin(k_x x)[C\sin(k_y y) + D\cos(k_y y)]\sin(k_z z),\\ E_z(x,y,z) &= \sin(k_x x)\sin(k_y y)[E\sin(k_z z) + F\cos(k_z z)],\end{aligned}$$

where $k_x = m\pi/a$. (Actually, there is no reason at this stage to assume that $k_x$, $k_y$, and $k_z$ are the *same* for all three components, and I should really affix a second subscript ($x$ for $E_x$, $y$ for $E_y$, and $z$ for $E_z$), but in a moment we shall see that *in fact* they *do* have to be the same, so to avoid cumbersome notation I'll *assume* they are from the start.)

Now $\nabla\cdot\mathbf{E} = 0 \Rightarrow k_x[A\cos(k_x x) - B\sin(k_x x)]\sin(k_y y)\sin(k_z z) + k_y\sin(k_x x)[C\cos(k_y y) - D\sin(k_y y)]\sin(k_z z) + k_z\sin(k_x x)\sin(k_y y)[E\cos(k_z z) - F\sin(k_z z)] = 0$. In particular, putting in $x = 0$, $k_x A\sin(k_y y)\sin(k_z z) = 0$, and hence $A = 0$. Likewise $y = 0 \Rightarrow C = 0$ and $z = 0 \Rightarrow E = 0$. (Moreover, if the $k$'s were *not* equal for different components, then by Fourier analysis this equation could not be satisfied (for all $x$, $y$, and $z$) unless the other three constants were *also* zero, and we'd be left with no field at all.) It follows that $-(Bk_x + Dk_y + Fk_z) = 0$ (in order that $\nabla \cdot \mathbf{E} = 0$), and we are left with

$$\boxed{\begin{array}{l}\mathbf{E} = B\cos(k_x x)\sin(k_y y)\sin(k_z z)\,\hat{\mathbf{x}} + D\sin(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{y}} + F\sin(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{z}},\\ \text{with } k_x = (m\pi/a),\ k_y = (n\pi/b),\ k_z = (l\pi/d)\ (l,\ m,\ n \text{ all integers}),\ \text{and } Bk_x + Dk_y + Fk_z = 0.\end{array}}$$

The corresponding magnetic field is given by $\mathbf{B} = -(i/\omega)\nabla \times \mathbf{E}$:

$$\begin{aligned}
B_x &= -\frac{i}{\omega}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{i}{\omega}\left[Fk_y \sin(k_x x)\cos(k_y y)\cos(k_z z) - Dk_z \sin(k_x x)\cos(k_y y)\cos(k_z z)\right],\\
B_y &= -\frac{i}{\omega}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{i}{\omega}\left[Bk_z \cos(k_x x)\sin(k_y y)\cos(k_z z) - Fk_x \cos(k_x x)\sin(k_y y)\cos(k_z z)\right],\\
B_z &= -\frac{i}{\omega}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\frac{i}{\omega}\left[Dk_x \cos(k_x x)\cos(k_y y)\sin(k_z z) - Bk_y \cos(k_x x)\cos(k_y y)\sin(k_z z)\right].
\end{aligned}$$

Or:

$$\boxed{\begin{aligned}\mathbf{B} &= -\frac{i}{\omega}(Fk_y - Dk_z)\sin(k_x x)\cos(k_y y)\cos(k_z z)\,\hat{\mathbf{x}} - \frac{i}{\omega}(Bk_z - Fk_x)\cos(k_x x)\sin(k_y y)\cos(k_z z)\,\hat{\mathbf{y}}\\ &- \frac{i}{\omega}(Dk_x - Bk_y)\cos(k_x x)\cos(k_y y)\sin(k_z z)\,\hat{\mathbf{z}}.\end{aligned}}$$

These *automatically* satisfy the boundary condition $B^{\perp} = 0$ ($B_x = 0$ at $x = 0$ and $x = a$, $B_y = 0$ at $y = 0$ and $y = b$, and $B_z = 0$ at $z = 0$ and $z = d$).

As a check, let's see if $\nabla \cdot \mathbf{B} = 0$ :

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= -\frac{i}{\omega}(Fk_y - Dk_z)k_x \cos(k_x x)\cos(k_y y)\cos(k_z z) - \frac{i}{\omega}(Bk_z - Fk_x)k_y \cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&- \frac{i}{\omega}(Dk_x - Bk_y)k_z \cos(k_x x)\cos(k_y y)\cos(k_z z)\\
&= -\frac{i}{\omega}(Fk_x k_y - Dk_x k_z + Bk_z k_y - Fk_x k_y + Dk_x k_z - Bk_y k_z)\cos(k_x x)\cos(k_y y)\cos(k_z z) = 0. \checkmark
\end{aligned}$$

The boxed equations satisfy all of Maxwell's equations, and they meet the boundary conditions. For TE modes, we pick $E_z = 0$, so $F = 0$ (and hence $Bk_x + Dk_y = 0$, leaving only the overall amplitude undetermined, for given $l$, $m$, and $n$); for TM modes we want $B_z = 0$ (so $Dk_x - Bk_y = 0$, again leaving only one amplitude undetermined, since $Bk_x + Dk_y + Fk_z = 0$). In either case ($\mathrm{TE}_{lmn}$ or $\mathrm{TM}_{lmn}$), the frequency is given by $\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = c^2\left[(m\pi/a)^2 + (n\pi/b)^2 + (l\pi/d)^2\right]$, or $\boxed{\omega = c\pi\sqrt{(m/a)^2 + (n/b)^2 + (l/d)^2}.}$

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**Problem 9.28**

Here $a = 2.28$ cm and $b = 1.01$ cm, so $\nu_{10} = \frac{1}{2\pi}\omega_{10} = \frac{c}{2a} = 0.66 \times 10^{10}$ Hz; $\nu_{20} = 2\frac{c}{2a} = 1.32 \times 10^{10}$ Hz; $\nu_{30} = 3\frac{c}{2a} = 1.97 \times 10^{10}$ Hz; $\nu_{01} = \frac{c}{2b} = 1.49 \times 10^{10}$ Hz; $\nu_{02} = 2\frac{c}{2b} = 2.97 \times 10^{10}$ Hz; $\nu_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1.62 \times 10^{10}$ Hz. Evidently just four modes occur: $\boxed{10,\ 20,\ 01,\ \text{and } 11.}$

To get only *one* mode you must drive the waveguide at a frequency between $\nu_{10}$ and $\nu_{20}$: $\boxed{0.66 \times 10^{10} < \nu < 1.32 \times 10^{10}\ \text{Hz.}}$ $\lambda = \frac{c}{\nu}$, so $\lambda_{10} = 2a$; $\lambda_{20} = a$. $\boxed{2.28\ \text{cm} < \lambda < 4.56\ \text{cm.}}$