TFY 4240 Løsning Øving 12

Problem 1

Problem 11.1

From Eq. 11.17, $\mathbf{A} = -\frac{\mu_0 p_0 \omega}{4\pi}\frac{1}{r}\sin[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$, so

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{1}{r}\sin[\omega(t-r/c)]\cos\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[-\sin^2\theta\frac{1}{r}\sin[\omega(t-r/c)]\right]\right\}\\
&= -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r^2}\left(\sin[\omega(t-r/c)] - \frac{\omega r}{c}\cos[\omega(t-r/c)]\right)\cos\theta - \frac{2\sin\theta\cos\theta}{r^2\sin\theta}\sin[\omega(t-r/c)]\right\}\\
&= \mu_0\epsilon_0\left\{\frac{p_0\omega}{4\pi\epsilon_0}\left(\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\cos\theta\right\}.
\end{aligned}$$

Meanwhile, from Eq. 11.12,

$$\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega^2}{c}\cos[\omega(t-r/c)] - \frac{\omega}{r}\sin[\omega(t-r/c)]\right\}\\
&= -\frac{p_0\omega}{4\pi\epsilon_0}\left\{\frac{1}{r^2}\sin[\omega(t-r/c)] + \frac{\omega}{rc}\cos[\omega(t-r/c)]\right\}\cos\theta. \quad \text{So } \nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}. \quad \text{qed}
\end{aligned}$$

Problem 11.3

$P = I^2R = q_0^2\omega^2\sin^2(\omega t)R$ (Eq. 11.15) $\Rightarrow \langle P\rangle = \frac{1}{2}q_0^2\omega^2 R$. Equate this to Eq. 11.22:

$$\frac{1}{2}q_0^2\omega^2 R = \frac{\mu_0 q_0^2 d^2\omega^4}{12\pi c} \Rightarrow \boxed{R = \frac{\mu_0 d^2\omega^2}{6\pi c}}; \text{ or, since } \omega = \frac{2\pi c}{\lambda},$$

$$R = \frac{\mu_0 d^2}{6\pi c}\frac{4\pi^2 c^2}{\lambda^2} = \frac{2}{3}\pi\mu_0 c\left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi(4\pi\times10^{-7})(3\times10^8)\left(\frac{d}{\lambda}\right)^2 = 80\pi^2\left(\frac{d}{\lambda}\right)^2\,\Omega = \boxed{789.6(d/\lambda)^2\,\Omega.}$$

For the wires in an ordinary radio, with $d = 5\times10^{-2}$ m and (say) $\lambda = 10^3$ m, $R = 790(5\times10^{-5})^2 = 2\times10^{-6}\,\Omega$, which is negligible compared to the Ohmic resistance.

Problem 11.4

By the superposition principle, we can *add* the potentials of the two dipoles. Let's first express V (Eq. 11.14) in Cartesian coordinates: $V(x, y, z, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{z}{x^2+y^2+z^2}\right)\sin[\omega(t-r/c)]$. That's for an oscillating dipole along the z axis. For one along x or y, we just change z to x or y. In the present case, $\mathbf{p} = p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t - \pi/2)\,\hat{\mathbf{y}}]$, so the one along y is delayed by a phase angle $\pi/2$: $\sin[\omega(t-r/c)] \to \sin[\omega(t-r/c) - \pi/2] = -\cos[\omega(t-r/c)]$ (just let $\omega t \to \omega t - \pi/2$). Thus

$$V = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\frac{x}{x^2+y^2+z^2}\sin[\omega(t-r/c)] - \frac{y}{x^2+y^2+z^2}\cos[\omega(t-r/c)]\right\}$$

$$= \boxed{-\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\sin\theta}{r}\{\cos\phi\sin[\omega(t-r/c)] - \sin\phi\cos[\omega(t-r/c)]\}.}$$

Similarly,

$$\mathbf{A} = \boxed{-\frac{\mu_0 p_0\omega}{4\pi r}\{\sin[\omega(t-r/c)]\,\hat{\mathbf{x}} - \cos[\omega(t-r/c)]\,\hat{\mathbf{y}}\}.}$$

We *could* get the fields by differentiating these potentials, but I prefer to work with Eqs. 11.18 and 11.19, using superposition. Since $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$, and $\cos\theta = z/r$, Eq. 11.18 can be written $\mathbf{E} = \frac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)]\left(\hat{\mathbf{z}} - \frac{z}{r}\hat{\mathbf{r}}\right)$. In the case of the rotating dipole, therefore,

$$\mathbf{E} = \boxed{\frac{\mu_0 p_0\omega^2}{4\pi r}\left\{\cos[\omega(t-r/c)]\left(\hat{\mathbf{x}} - \frac{x}{r}\hat{\mathbf{r}}\right) + \sin[\omega(t-r/c)]\left(\hat{\mathbf{y}} - \frac{y}{r}\hat{\mathbf{r}}\right)\right\},}$$

$$\mathbf{B} = \boxed{\frac{1}{c}(\hat{\mathbf{r}}\times\mathbf{E}).}$$

$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E}\times(\hat{\mathbf{r}}\times\mathbf{E})] = \frac{1}{\mu_0 c}\left[E^2\,\hat{\mathbf{r}} - (\mathbf{E}\cdot\hat{\mathbf{r}})\mathbf{E}\right] = \frac{E^2}{\mu_0 c}\hat{\mathbf{r}}$ (notice that $\mathbf{E}\cdot\hat{\mathbf{r}} = 0$). Now

$$E^2 = \left(\frac{\mu_0 p_0\omega^2}{4\pi r}\right)^2\left\{a^2\cos^2[\omega(t-r/c)] + b^2\sin^2[\omega(t-r/c)] + 2(\mathbf{a}\cdot\mathbf{b})\sin[\omega(t-r/c)]\cos[\omega(t-r/c)]\right\},$$

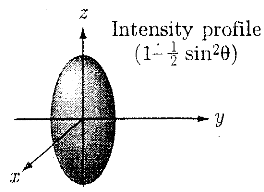
where $\mathbf{a} \equiv \hat{\mathbf{x}} - (x/r)\hat{\mathbf{r}}$ and $\mathbf{b} \equiv \hat{\mathbf{y}} - (y/r)\hat{\mathbf{r}}$. Noting that $\hat{\mathbf{x}}\cdot\mathbf{r} = x$ and $\hat{\mathbf{y}}\cdot\mathbf{r} = y$, we have

$$a^2 = 1 + \frac{x^2}{r^2} - 2\frac{x^2}{r^2} = 1 - \frac{x^2}{r^2};\ b^2 = 1 - \frac{y^2}{r^2};\ \mathbf{a}\cdot\mathbf{b} = -\frac{y}{r}\frac{x}{r} - \frac{x}{r}\frac{y}{r} + \frac{xy}{r^2} = -\frac{xy}{r^2}.$$

$$\begin{aligned}
E^2 &= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{\left(1 - \frac{x^2}{r^2}\right)\cos^2[\omega(t-r/c)] + \left(1 - \frac{y^2}{r^2}\right)\sin^2[\omega(t-r/c)] \right. \\
&\quad \left. - 2\frac{xy}{r^2}\sin[\omega(t-r/c)]\cos[\omega(t-r/c)]\right\} \\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \frac{1}{r^2}\left(x^2\cos^2[\omega(t-r/c)] + 2xy\sin[\omega(t-r/c)]\cos[\omega(t-r/c)] + y^2\sin^2[\omega(t-r/c)]\right)\right\} \\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \frac{1}{r^2}\left(x\cos[\omega(t-r/c)] + y\sin[\omega(t-r/c)]\right)^2\right\}
\end{aligned}$$

But $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$.

$$\begin{aligned}
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \sin^2\theta\left(\cos\phi\cos[\omega(t-r/c)] + \sin\phi\sin[\omega(t-r/c)]\right)^2\right\} \\
&= \left(\frac{\mu_0 p_0 \omega^2}{4\pi r}\right)^2 \left\{1 - \left(\sin\theta\cos[\omega(t-r/c)-\phi]\right)^2\right\}.
\end{aligned}$$



$$\mathbf{S} = \boxed{\frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left\{1 - \left(\sin\theta\cos[\omega(t-r/c)-\phi]\right)^2\right\}\hat{\mathbf{r}}.}$$

$$\langle\mathbf{S}\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi r}\right)^2 \left[1 - \frac{1}{2}\sin^2\theta\right]\hat{\mathbf{r}}.$$

$$\begin{aligned}
P &= \int \langle\mathbf{S}\rangle\cdot d\mathbf{a} = \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\right)^2 \int \frac{1}{r^2}\left(1 - \frac{1}{2}\sin^2\theta\right) r^2\sin\theta\, d\theta\, d\phi \\
&= \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}2\pi\left[\int_0^\pi \sin\theta\, d\theta - \frac{1}{2}\int_0^\pi \sin^3\theta\, d\theta\right] = \frac{\mu_0 p_0^2\omega^4}{8\pi c}\left(2 - \frac{1}{2}\cdot\frac{4}{3}\right) = \boxed{\frac{\mu_0 p_0^2\omega^4}{6\pi c}.}
\end{aligned}$$

This is *twice* the power radiated by either oscillating dipole alone (Eq. 11.22). In general, $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}[(\mathbf{E}_1 + \mathbf{E}_2)\times(\mathbf{B}_1 + \mathbf{B}_2)] = \frac{1}{\mu_0}[(\mathbf{E}_1\times\mathbf{B}_1) + (\mathbf{E}_2\times\mathbf{B}_2) + (\mathbf{E}_1\times\mathbf{B}_2) + (\mathbf{E}_2\times\mathbf{B}_1)] = \mathbf{S}_1 + \mathbf{S}_2 +$ cross terms. In this particular case, the fields of 1 and 2 are 90° out of phase, so the cross terms go to zero in the time averaging, and the total power radiated is just the sum of the two individual powers.

Problem 11.8

$\mathbf{p}(t) = p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] \Rightarrow \ddot{\mathbf{p}}(t) = -\omega^2 p_0[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] \Rightarrow$

$[\ddot{\mathbf{p}}(t)]^2 = \omega^4 p_0^2[\cos^2(\omega t) + \sin^2(\omega t)] = p_0^2\omega^4$. So Eq. 11.59 says $\boxed{\mathbf{S} = \frac{\mu_0 p_0^2\omega^4}{16\pi^2 c}\frac{\sin^2\theta}{r^2}\hat{\mathbf{r}}.}$ (This appears to disagree with the answer to Prob. 11.4. The reason is that in Eq. 11.59 the polar axis is along the direction of $\ddot{\mathbf{p}}(t_0)$; as the dipole rotates, so do the axes. Thus the angle θ here is not the same as in Prob. 11.4.) Meanwhile, Eq. 11.60 says $\boxed{P = \frac{\mu_0 p_0^2\omega^4}{6\pi c}.}$ (This *does* agree with Prob. 11.4, because we have now integrated over all angles, and the orientation of the polar axis irrelevant.)