ØVING 13

LØSNINGSFORSLAG.

Oppgave 1.

The relativistic generalisation of Larmour's formula reads:

$$P = \frac{\mu_0 q^2}{6\pi c m^2}\left(\frac{dp_\mu}{d\tau}\cdot\frac{dp^\mu}{d\tau}\right) \qquad \mu = 1, \dots 4.$$

Sum over equal indicies.

$$p^\mu = \left(\frac{E}{c}, \bar{p}\right)$$

$$E = \gamma m c^2 \qquad \bar{p} = \gamma m \bar{v} \qquad , d\tau = \frac{dt}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\frac{dp^\mu}{d\tau}\cdot\frac{dp_\mu}{d\tau} = -\frac{1}{c^2}\left(\frac{dE}{d\tau}\right)^2 + \left(\frac{d\bar{p}}{d\tau}\right)^2$$

We have to calculate $\frac{dE}{d\tau}$ and $\frac{d\bar{p}}{d\tau}$

$$\frac{dE}{d\tau} = \frac{dE}{dp}\cdot\frac{dp}{d\tau}$$

$$\frac{dE}{dp} = \frac{d}{dp}\sqrt{m^2c^4 + p^2c^2} = \frac{pc^2}{E} = \frac{p}{\gamma m} = v$$

$$\frac{d\bar{p}}{d\tau} = \gamma\,\frac{d\bar{p}}{dt}$$

$$\frac{d\bar{p}}{dt} = \frac{d}{dt}\gamma m \bar{v} = \frac{d}{dt}\,\frac{m\bar{v}}{\sqrt{1 - v^2/c^2}}$$

$$\frac{d\bar{p}}{dt} = \gamma m \dot{\bar{v}} + m\bar{v} \; \frac{\bar{v}\cdot\dot{\bar{v}}}{c^2}\gamma^3$$

$$= \gamma mc\,(\dot{\bar{\beta}} + \bar{\beta}\,(\bar{\beta}\cdot\dot{\bar{\beta}})\gamma^2)$$

$$\frac{dp_\mu}{dt}\cdot\frac{dp^\mu}{dt} = -\frac{v^2}{c^2}\left(\frac{dp}{dt}\right)^2 + \left(\frac{d\bar{p}}{dt}\right)^2$$

$$= -\beta^2\left(\frac{dp}{dt}\right)^2 + \left(\frac{d\bar{p}}{dt}\right)^2$$

$$\beta\left(\frac{dp}{dt}\right) = \beta\frac{d}{dt}\gamma m v = \beta\gamma\frac{d}{dt}\gamma m v$$

Next: $\frac{dv}{dt} = \frac{d}{dt}\sqrt{v_i v_i} = \frac{v_i\dot{v}_i}{v} = \frac{\bar{v}\cdot\dot{\bar{v}}}{v}$

Thus $\beta\frac{dp}{dt} = \beta\gamma m\left(\gamma\frac{\bar{v}\cdot\dot{\bar{v}}}{v} + \gamma^3 v\frac{\bar{v}\cdot\dot{\bar{v}}}{c^2}\right)$

$$= \gamma^2 mc\,(\bar{\beta}\cdot\dot{\bar{\beta}} + \gamma^2\beta^2\;\bar{\beta}\cdot\dot{\bar{\beta}})$$

$$= \gamma^2 mc(\bar{\beta}\cdot\dot{\bar{\beta}})(1+\gamma^2\beta^2)$$

$$= \gamma^4 mc\;\bar{\beta}\cdot\dot{\bar{\beta}}$$

$$\frac{d\bar{p}}{dt} = \gamma^4 mc\left(\frac{\dot{\bar{\beta}}}{\gamma^2} + \bar{\beta}(\bar{\beta}\cdot\dot{\bar{\beta}})\right)$$

$$= \gamma^4 mc\,(\dot{\bar{\beta}}(1-\beta^2) + \bar{\beta}\,(\bar{\beta}\cdot\dot{\bar{\beta}}))$$

Next:

$$\frac{dp_\mu}{dt}\frac{dp^\mu}{dt} = \gamma^8 m^2c^2\Big(-(\bar\beta\cdot\dot{\bar\beta})^2 + (1-\beta^2)^2\dot{\bar\beta}^2$$
$$+2(1-\beta^2)(\bar\beta\cdot\dot{\bar\beta})^2 + \beta^2(\bar\beta\cdot\dot{\bar\beta})^2\Big)$$

$$= \gamma^8 m^2c^2\left((1-\beta^2)^2\dot{\bar\beta}^2 + (1-\beta^2)(\bar\beta\cdot\dot{\bar\beta})^2\right)$$

$$= \gamma^8 m^2c^2\left[\left((1-\beta^2)\dot{\bar\beta}^2 + (\bar\beta\cdot\dot{\bar\beta})^2\right)\right](1-\beta^2)$$

$$= \gamma^6 m^2c^2\left[(1-\beta^2)\dot{\bar\beta}^2 + (\bar\beta\cdot\dot{\bar\beta})^2\right]$$

Call the angle between $\bar\beta$ and $\dot{\bar\beta}$ for Θ

$$\Rightarrow \gamma^6 m^2c^2\left[\dot\beta^2 - \beta^2\dot{\bar\beta}^2 + \beta^2\dot{\bar\beta}^2\cos^2\Theta\right]$$

$$= \gamma^6 m^2c^2\left[\dot\beta^2 - \bar\beta^2\dot{\bar\beta}^2\sin^2\Theta\right]$$

This can be written as

$$\gamma^6 m^2c^2\left[\dot\beta^2 - (\bar\beta\times\dot{\bar\beta})^2\right]$$

So finally

$$P = \frac{\mu_0 q^2 c}{6\pi}\left[\dot\beta^2 - (\bar\beta\times\dot{\bar\beta})^2\right]$$

Problem 12.52

$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$. Differentiate: $\partial_\mu\partial_\nu F^{\mu\nu} = \mu_0 \partial_\mu J^\mu$.

But $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$ (the combination is *symmetric*) while $F^{\nu\mu} = -F^{\mu\nu}$ (*anti*symmetric).

$\therefore\ \partial_\mu\partial_\nu F^{\mu\nu} = 0$. [Why? Well, these indices are both summed from $0 \to 3$, so it doesn't matter which we call μ, which ν: $\partial_\mu\partial_\nu F^{\mu\nu} = \partial_\nu\partial_\mu F^{\nu\mu} = \partial_\mu\partial_\nu(-F^{\mu\nu}) = -\partial_\mu\partial_\nu F^{\mu\nu}$. But if a quantity is equal to minus itself, it must be zero.] *Conclusion:* $\partial_\mu J^\mu = 0$. qed

Problem 12.53

We know that $\partial_\nu G^{\mu\nu} = 0$ is equivalent to the two homogeneous Maxwell equations, $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$. All we have to show, then, is that $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ is *also* equivalent to them. Now this equation stands for 64 separate equations ($\mu = 0 \to 3$, $\nu = 0 \to 3$, $\lambda = 0 \to 3$, and $4\times 4\times 4 = 64$). But many of them are redundant, or trivial.

Suppose two indices are the same (say, $\mu = \nu$). Then $\partial_\lambda F_{\mu\mu} + \partial_\mu F_{\mu\lambda} + \partial_\mu F_{\lambda\mu} = 0$. But $F_{\mu\mu} = 0$ and $F_{\mu\lambda} = -F_{\lambda\mu}$, so this is trivial: $0 = 0$. To get anything significant, then, μ, ν, λ must all be *different*. They could be *all spatial* ($\mu, \nu, \lambda = 1, 2, 3 = x, y, z$ — or some permutation thereof), or *one temporal* and *two spatial* ($\mu = 0$, $\nu, \lambda = 1, 2$ or $2, 3$, or $1, 3$ — or some permutation). Let's examine these two cases separately.

All spatial: say, $\mu = 1$, $\nu = 2$, $\lambda = 3$ (other permutations yield the same equation, or minus it).

$$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = 0 \Rightarrow \frac{\partial}{\partial z}(B_z) + \frac{\partial}{\partial x}(B_x) + \frac{\partial}{\partial y}(B_y) = 0 \Rightarrow \nabla\cdot\mathbf{B} = 0.$$

One temporal: say, $\mu = 0$, $\nu = 1$, $\lambda = 2$ (other permutations of these indices yield the same result, or minus it).

$$\partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0 \Rightarrow \frac{\partial}{\partial y}\left(-\frac{E_x}{c}\right) + \frac{\partial}{\partial(ct)}(B_z) + \frac{\partial}{\partial x}\left(\frac{E_y}{c}\right) = 0,$$

or $-\frac{\partial B_z}{\partial t} + \left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right) = 0$, which is the z component of $-\frac{\partial\mathbf{B}}{\partial t} = \nabla\times\mathbf{E}$. (If $\mu = 0$, $\nu = 1$, $\lambda = 2$, we get the y component; for $\nu = 2$, $\lambda = 3$ we get the x component.)

Conclusion: $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ is equivalent to $\nabla\cdot\mathbf{B} = 0$ and $\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$, and hence to $\partial_\nu G^{\mu\nu} = 0$. qed

Problem 12.54

$K^0 = q\eta_\nu F^{0\nu} = q(\eta_1 F^{01} + \eta_2 F^{02} + \eta_3 F^{03}) = q(\boldsymbol{\eta}\cdot\mathbf{E})/c = \boxed{\frac{q}{c}\gamma\mathbf{u}\cdot\mathbf{E}.}$ Now from Eq. 12.71 we know that $K^0 = \frac{1}{c}\frac{dW}{d\tau}$, where W is the energy of the particle. Since $d\tau = \frac{1}{\gamma}dt$, we have:

$$\frac{1}{c}\gamma\frac{dW}{dt} = \frac{q}{c}\gamma(\mathbf{u}\cdot\mathbf{E}) \Rightarrow \boxed{\frac{dW}{dt} = q(\mathbf{u}\cdot\mathbf{E}).}$$

This says *the power delivered to the particle is force* ($q\mathbf{E}$) *times velocity* ($\mathbf{u}$) — which is as it *should* be.