LØSNING

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I

Feltlikning $\quad -\partial_\mu \dfrac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = \underline{\Box^2 \phi = 0}$

$$\underline{S_{\mu\nu}} = \mathcal{L}\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}}\phi_{,\mu} = \underline{\mathcal{L}\delta_{\mu\nu} + \phi_{,\mu}\phi_{,\nu}}$$

$$\underline{M} = \int (\underline{r} \times \underline{g})\, d^3x$$

Utvider til 4 komponenter: $M_{\mu\nu} = \int (x_\mu g_\nu - x_\nu g_\mu) d^3x$
og innfører $M_{\mu\nu\rho} = x_\mu S_{\nu\rho} - x_\nu S_{\mu\rho}$.

Da $\partial_\rho M_{\mu\nu\rho} = S_{\nu\mu} - S_{\mu\nu} = 0$ er

$$\frac{d}{dx_4}\int M_{\mu\nu 4}\, d^3x = \frac{1}{ic}\frac{d}{dt}\int (x_\mu S_{\nu 4} - x_\nu S_{\mu 4})\, d^3x = 0$$

$$\Rightarrow \quad \underline{\frac{d}{dt}\underline{M} = 0}$$

Sylindersymmetri om $x_1$-aksen: $\quad (\rho = \sqrt{x_2^2 + x_3^2})$

$$\underline{M_1} = \int (x_2 g_3 - x_3 g_2)\, d^3x = -\frac{1}{c^2}\int \dot{\phi}\,[x_2\phi_{,3} - x_3\phi_{,2}]\, d^3x$$

$$= -\frac{1}{c^2}\int \dot{\phi}\left[x_2 \frac{\partial \phi}{\partial \rho}\frac{x_3}{\rho} - x_3\frac{\partial \phi}{\partial \rho}\frac{x_2}{\rho}\right] d^3x = \underline{0}$$

Monokromatisk bølge. Da $\Box^2\phi = 0$ er $\underline{k}^2 = \dfrac{\omega^2}{c^2}$

$$\underline{S} = -\dot{\phi}\nabla\phi, \quad h = -S_{44} = \frac{1}{2}\left[(\nabla\phi)^2 + \frac{1}{c^2}\dot{\phi}^2\right]$$

$$\Rightarrow \quad \underline{u} = \underline{S}/h = \frac{A^2\omega\, \underline{k} \sin^2\varphi}{\frac{1}{2}A^2(\underline{k}^2 + \frac{\omega^2}{c^2}\sin^2\varphi)} = c\hat{\underline{k}}\,; \quad \underline{u = c}$$

$$\varphi = \underline{k}\cdot\underline{r} - \omega t.$$

## II

a)

$$\mathcal{L}-\mathcal{L}' = -\tfrac{1}{2} A_{\mu,\nu} A_{\mu,\nu} + \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} + \tfrac{1}{2} A_{\mu,\mu} A_{\nu,\nu}$$

$$= \tfrac{1}{2}\left(A_{\mu,\mu} A_{\nu,\nu} - A_{\mu,\nu} A_{\nu,\mu}\right) = \tfrac{1}{2}\partial_\mu\left(A_\mu A_{\nu,\nu} - A_\nu A_{\mu,\nu}\right)$$

Da $(\mathcal{L}-\mathcal{L}')$ er en fullstendig divergens er $\mathcal{L}$ og $\mathcal{L}'$ ekvivalente.

b) Avstanden måles ved tiden $t$. Innfører lokalt inertialsystem $\overset{\circ}{I}$ med Galileiske koordinater $X^\mu$.

$$dl^2 = dX^i dX^i \quad ; \quad dX^i = \frac{\partial X^i}{\partial x^\nu} dx^\nu = \frac{\partial X^i}{\partial x^k} dx^k \text{ fordi } dx^4 = 0.$$

$$\Rightarrow \quad dl^2 = \frac{\partial X^i}{\partial x^k} \frac{\partial X^i}{\partial x^l} dx^k dx^l.$$

Da $ds^2 = G_{\mu\nu} dX^\mu dX^\nu = g_{\alpha\beta} dx^\alpha dx^\beta$ er

$$g_{\alpha\beta} = G_{\mu\nu} \frac{\partial X^\mu}{\partial x^\alpha} \frac{\partial X^\nu}{\partial x^\beta} = \frac{\partial X^i}{\partial x^\alpha} \frac{\partial X^i}{\partial x^\beta} - \frac{\partial X^4}{\partial x^\alpha} \frac{\partial X^4}{\partial x^\beta}.$$

For et av punktene er $dX^i = \frac{\partial X^i}{\partial x^\nu} dx^\nu = \frac{\partial X^i}{\partial x^4} dx^4$ da

punktet har konstante romkoordinater $(dx^i = 0)$. Da $\overset{\circ}{I}$ er et momentant hvilesystem for punktet er hastigheten null:

$$c\frac{dX^i}{dX^4} = c\frac{\partial X^i}{\partial x^4}\frac{dx^4}{dX^4} = 0 \quad \Rightarrow \quad \frac{\partial X^i}{\partial x^4} = 0.$$

Altså er $g_{j4} = -\frac{\partial X^4}{\partial x^j} \frac{\partial X^4}{\partial x^4} \Rightarrow$

$$\underline{dl^2} = \left(g_{jk} - \frac{g_{j4} g_{k4}}{g_{44}}\right) dx^j dx^k = \underline{\gamma_{jk} dx^j dx^k.}$$

III

Lar $Z$-aksen være rotasjonsakse.

$ds^2 = dR^2 + R^2 d\Theta^2 + dZ^2 - c^2 dT^2$, der $X^\mu = (R, \Theta, Z, cT)$

er koordinatene i det faste inertialsystemet I.

I K innføres $x^\mu = (r, \vartheta, z, ct)$, der

$R = r, \ \Theta = \vartheta + \omega t, \ Z = z, \ T = t$.

Altså

$$ds^2 = dr^2 + r^2(d\vartheta + \omega\, dt)^2 + dz^2 - c^2 dt^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$\Rightarrow$

$g_{11} = 1, \ g_{22} = r^2, \ g_{33} = 1, \ g_{44} = -(1 - r^2\omega^2/c^2), \ g_{24} = \frac{\omega r^2}{c}$

Lysfrekvens i K: $\nu_1 = \frac{d\tau}{dt}\mathring{\nu} = \mathring{\nu}\sqrt{-g_{44}} = \mathring{\nu}\sqrt{1 - r^2\omega^2/c^2}$.

Lyshastigheten i K i emisjonsøyeblikket: $w = \sqrt{\gamma_{jk}\, dx^j dx^k} = dl/dt$.

$w = \sqrt{\gamma_{22}\, \dot{\vartheta}^2} = \sqrt{\gamma_{22}}\, \dot{\vartheta}$ fordi hast. er tangential.

Finner $\dot{\vartheta}$ ut fra likn. $ds^2 = 0 \Rightarrow$

$g_{22}(dx^2)^2 + 2g_{24}\, dx^2 dx^4 + g_{44}(dx^4)^2 = 0$

$$\Rightarrow \dot{\vartheta} = \frac{-\omega r^2 + \sqrt{\omega^2 r^4 + r^2(1 - r^2\omega^2/c^2)c^2}}{r^2} = \frac{c}{r}\left(1 - \frac{\omega r}{c}\right)$$

$$\gamma_{22} = g_{22} - \frac{g_{24}^2}{g_{44}} = \frac{r^2}{1 - r^2\omega^2/c^2}$$

$$\Rightarrow \quad w = c\sqrt{\frac{1 - r\omega/c}{1 + r\omega/c}}$$

$\dot{\vartheta}$ kan også bestemmes ut fra $c = \frac{R\, d\Theta}{dT} = r\dot{\Theta}$

$= r(\dot{\vartheta} + \omega) \Rightarrow r\dot{\vartheta} = c - r\omega$, som ovenfor.