Teoretisk fysikk - Videreg. kurs A - Feltmekanikk 23.1.86

Løsninger:

I a) Varierer feltet med faste verdier på grensene:

$$\delta S = \int\left[\frac{\partial \mathcal{L}}{\partial \psi_a}\,\delta\psi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)}\,\delta(\partial_\mu\psi_a)\right] d^4x = \int_{\text{overflate}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi_a)}\,\delta\psi_a\right]_{x_{\mu 1}}^{x_{\mu 2}} d^3x + \int\left(\frac{\partial \mathcal{L}}{\partial \psi_a} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi_a)}\right)\delta\psi_a = 0$$

$$\Rightarrow \quad \underline{\frac{\partial \mathcal{L}}{\partial \psi_a} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_a} = 0} \qquad a = 1, 2 \dots n$$

b) To feltkomponenter ψ og $\bar\psi$: $\mathcal{L}(\psi, \bar\psi, \partial_\mu\psi, \partial_\mu\bar\psi, x^\mu)$

$$\mathcal{L} = \tfrac{1}{2}\bar\psi\, c\left[\gamma^\mu\left(\tfrac{\hbar}{i}\partial_\mu - eA_\mu\right)\psi - mc\,\psi\right] + \tfrac{1}{2} c\left[\left(-\tfrac{\hbar}{i}\partial_\mu - eA_\mu\right)\bar\psi\gamma^\mu - mc\,\bar\psi\right]\psi$$

Felt likning for ψ:

Varierer $\bar\psi$. Trenger:

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} = \tfrac{1}{2} c\left[\gamma^\mu\left(\tfrac{\hbar}{i}\partial_\mu - eA_\mu\right) - mc\right]\psi + \tfrac{1}{2}\left[-eA_\mu\gamma^\mu - mc\right]\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)} = \tfrac{1}{2} c\left(-\tfrac{\hbar}{i}\gamma^\mu\psi\right)$$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)} = \underline{c\left[\gamma^\mu\left(\tfrac{\hbar}{i}\partial_\mu - eA_\mu\right) - mc\right]\psi = 0} \qquad \text{Dirac-likningen}$$

c) Bevarelsessetning for energitetthet:

Fra
$$\frac{d\mathcal{L}}{dt} = \frac{\partial \mathcal{L}}{\partial t} + \frac{\partial \mathcal{L}}{\partial \psi}\frac{\partial \psi}{\partial t} + \frac{\partial\bar\psi}{\partial t}\frac{\partial \mathcal{L}}{\partial\bar\psi} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial(\partial_\mu\psi)}{\partial t} + \frac{\partial(\partial_\mu\bar\psi)}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)}$$

Omflytting og bruk av Euler-Lagrange likningen gir:

$$\frac{\partial \mathcal{L}}{\partial t} = \frac{d\mathcal{L}}{dt} - \left(\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\right)\frac{\partial\psi}{\partial t} - \frac{\partial\bar\psi}{\partial t}\left(\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)}\right) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\left(\frac{\partial}{\partial x^\mu}\frac{\partial\psi}{\partial t}\right) - \left(\frac{\partial}{\partial x^\mu}\frac{\partial\bar\psi}{\partial t}\right)\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)}$$

$$= \frac{d}{dt}\mathcal{L} - \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\frac{\partial\psi}{\partial t} + \frac{\partial\bar\psi}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar\psi)}\right)$$

$$= \frac{d}{dt}\left\{\mathcal{L} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\frac{\partial\psi}{\partial x^0} + \frac{\partial\bar\psi}{\partial x^0}\frac{\partial \mathcal{L}}{\partial(\partial_0\bar\psi)}\right)\right\} - \frac{\partial}{\partial x^k}\left(\frac{\partial \mathcal{L}}{\partial(\partial_k\psi)}\frac{\partial\psi}{\partial t} + \frac{\partial\bar\psi}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_k\bar\psi)}\right)$$

$$\equiv -\left[\frac{d}{dt}T^0{}_0 + \frac{d}{dx^k}T^k{}_0\right] = 0 \qquad \text{når } \frac{\partial \mathcal{L}}{\partial t} = 0.$$

Med $\mathcal{H} = \text{konst}\; T^0{}_0$ og $S^k = \text{konst}\; c\,T^k{}_0$

gir dette
$$\underline{\frac{d}{dt}\mathcal{H} + \nabla\cdot\vec{S} = 0}$$

med
$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\partial_0\psi + \partial_0\bar\psi\frac{\partial \mathcal{L}}{\partial(\partial_0\bar\psi)} - \mathcal{L} \qquad \text{energitetthet}$$

$$S^k = \frac{\partial \mathcal{L}}{\partial(\partial_k\psi)}\frac{\partial\psi}{\partial t} + \frac{\partial\bar\psi}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_k\bar\psi)} \qquad \text{energistrømtetthet.}$$

T. F. Vid. A Feltmekanikk 23. 76

I c fortsatt

Trenger $\quad \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{1}{2} c \frac{\hbar}{i} \bar{\psi} \gamma^\mu$

Innsatt:

$$\mathcal{H} = \tfrac{1}{2} c\, \bar{\psi}\, \gamma^0 \tfrac{\hbar}{i} \partial_0 \psi + \tfrac{1}{2} c\, \partial_0 \bar{\psi}\, \gamma^0 (-\tfrac{\hbar}{i}) \psi - \Big(\tfrac{1}{2} c\, \bar{\psi} (\gamma^\mu \tfrac{\hbar}{i} \partial_\mu - e A_\mu) \psi + \tfrac{1}{2} c \big(\partial_\mu \bar{\psi} \gamma^\mu (\tfrac{\hbar}{i}) - e \bar{\psi} \gamma^\mu A_\mu \psi\big) - mc^2 \bar{\psi}\psi\Big)$$

$$= -\tfrac{1}{2} c\, \bar{\psi} (\gamma^k \tfrac{\hbar}{i} \partial_k - e A_k) \psi - \tfrac{1}{2} c (-\tfrac{\hbar}{i} \partial_k - e A_k) \bar{\psi} \gamma^k \psi + c e \bar{\psi} \gamma^0 A_0 \psi + mc^2 \bar{\psi}\psi$$

$$\underline{= \tfrac{1}{2} c \big[\bar{\psi}\, \vec{\gamma} (i\hbar \vec{\nabla} - e\vec{A}) \psi + ((-i\hbar \vec{\nabla} - e\vec{A}) \bar{\psi}) \vec{\gamma} \psi\big] + ce\, \bar{\psi}\, \gamma^0 \tfrac{\Phi}{c} \psi + mc^2 \bar{\psi}\psi}$$

når en har benyttet $\quad \gamma^\mu A_\mu = \gamma^0 A_0 + \gamma^k A_k = \gamma^0 A^0 - \gamma^k A^k = \gamma^0 \frac{\Phi}{c} - \vec{\gamma} \vec{A}$

$$\gamma^\mu \partial_\mu = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^0 \frac{\partial}{\partial x^0} + \gamma^k \frac{\partial}{\partial x^k} = \frac{\gamma^0}{c} \frac{\partial}{\partial t} + \vec{\gamma} \vec{\nabla}$$

$$\underline{S^k = \tfrac{1}{2} \tfrac{\hbar c}{i} (\bar{\psi} \gamma^k \dot{\psi} - \dot{\bar{\psi}} \gamma^k \psi)}$$

Ser også at vi kan skrive. Denne divergensen gir 0 bidrag til 4-dimensjon.

$$\mathcal{H} = \mathcal{H}' - \partial_k \Big(\bar{\psi}\, \frac{i\hbar c}{2} \gamma^k \psi\Big) \quad \text{med} \quad \mathcal{H}' = c\, \bar{\psi}\, \vec{\gamma} (i\hbar \nabla - e\vec{A}) \psi + mc^2 \bar{\psi}\psi + \bar{\psi} \gamma^0 e \Phi \psi$$

$$S^k = S^{k\prime} + \frac{\partial}{\partial t} \Big(\bar{\psi}\, \frac{i\hbar c}{2} \gamma^k \psi\Big) \quad \text{med} \quad S^{k\prime} = \frac{\hbar c}{i} \bar{\psi} \gamma^k \dot{\psi}$$

og altså $\quad \dfrac{\partial \mathcal{H}'}{\partial t} + \nabla \vec{S}' = 0$

II a) Varierer banen $x^k \to x^k + \delta x^k$ med faste endepunkt og får da $\quad S = \int_1^2 L\, dt$

$$\delta S = \int \Big[\frac{\partial L}{\partial x^k} \delta x^k + \frac{\partial L}{\partial \dot{x}^k} \delta \dot{x}^k\Big] dt = \int \Big(\frac{\partial L}{\partial x^k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^k}\Big) \delta x^k\, dt = 0$$

som gir $\quad \underline{\dfrac{\partial L}{\partial x^k} - \dfrac{d}{dt} \dfrac{\partial L}{\partial \dot{x}^k} = 0}$

b) Den lille partikkelen går slik at

$\int ds$ = ekstremum $\quad \delta \int_1^2 ds \Leftrightarrow \delta \int_1^2 \frac{ds}{dt} dt = 0$

$$ds = \frac{ds}{dt} dt = \sqrt{\Big(1 - \frac{\varepsilon}{r}\Big) c^2 - \Big(1 - \frac{\varepsilon}{r}\Big)^{-1} \Big(\frac{dr}{dt}\Big)^2 - r^2 \Big[\Big(\frac{d\theta}{dt}\Big)^2 + \sin^2\theta \Big(\frac{d\phi}{dt}\Big)^2\Big]}\; dt$$

Setter $\quad L = -mc \dfrac{ds}{dt} \quad$ og samme virkningsintegral som i a)

$$S = -mc^2 \int_{t_1}^{t_2} \sqrt{1 - \frac{\varepsilon}{r} - \frac{1}{c^2} \Big[\Big(1 + \frac{\varepsilon}{r} + \frac{\varepsilon^2}{r^2} + \dots\Big) \Big(\frac{dr}{dt}\Big)^2 + r^2 \Big(\frac{d\theta}{dt}\Big)^2 + \sin^2\theta \Big(\frac{d\phi}{dt}\Big)^2\Big]}\; dt$$

$$= -mc^2 \int_{t_1} dt \Big(1 - \frac{1}{2} \frac{\varepsilon}{r} - \frac{1}{2c^2} \Big[\Big(\frac{dr}{dt}\Big)^2 + r^2 \Big(\frac{d\theta}{dt}\Big)^2 + r^2 \sin^2\theta \Big(\frac{d\phi}{dt}\Big)^2\Big] + \mathcal{O}\Big(\varepsilon^2, \varepsilon \Big(\frac{v}{c}\Big)^2\Big)\Big)$$

$$= -mc^2 (t_2 - t_1) + \int_{t_1}^{t_2} \Big[\frac{1}{2} m v^2 + \frac{mc^2 \varepsilon}{2r}\Big] dt + \dots \qquad v^2 = \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$

TF Vidg. A Feltmek. 23.1.86

II b fortsatt

Dette svarer til en Lagrange-funksjon $L = T - V = \frac{1}{2}mv^2 - V$

med $V = -\frac{mc^2\varepsilon}{2r}$

som gir bevegelsesligningen

$$m\ddot{\vec r} = -\nabla V = -\frac{mc^2\varepsilon}{2r^2}\hat{e}_r$$

Spesielt blir radialligningen

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot r} = 0 \Rightarrow \frac{d}{dt}(m\dot r) = -\frac{mc^2\varepsilon}{2r^2} + mr(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) = -\frac{mc^2\varepsilon}{2r^2} + \frac{\ell^2}{mr^3}$$

med dreieimpuls $\ell^2 = m^2 r^4(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) = \text{konst.}$

I det newtonske gravitasjonsfeltet er $V = -\frac{GMm}{r}$

som gir $\varepsilon = \frac{2GM}{c^2}$ $\qquad m\ddot{\vec r} = -\frac{GMm}{r^2}\hat{e}_r$

c) For $d\vec{x} = 0$ $\quad dr = 0,\ d\theta = 0,\ d\phi = 0$

$$\Delta\tau = \sqrt{\frac{\Delta s^2}{c^2}} = \sqrt{1-\frac{\varepsilon}{r}}\,\Delta t$$

$\Delta\tau$ minker når r minker (selvsagt $r > \varepsilon$)

Sender $\Delta\tau_1 = \sqrt{1-\frac{\varepsilon}{r_1}}\,\Delta t_1$

Mottaker $\Delta\tau_2 = \sqrt{1-\frac{\varepsilon}{r_2}}\,\Delta t_2$

$\Delta t_1 = \Delta t_2$ gir $\frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{1-\frac{\varepsilon}{r_1}}{1-\frac{\varepsilon}{r_2}}}$

Mottaker frekvens: (Når $\Delta\tau_1$ og $\Delta\tau_2$ er svingetiden for en svingning)

$$\nu_2 = \frac{1}{\Delta\tau_2} = \frac{1}{\Delta\tau_1}\frac{\Delta\tau_1}{\Delta\tau_2} = \nu_1\sqrt{\frac{1-\frac{\varepsilon}{r_1}}{1-\frac{\varepsilon}{r_2}}} \approx \nu_1\left(1-\frac{1}{2}\frac{\varepsilon}{r_1}\right)\left(1+\frac{1}{2}\frac{\varepsilon}{r_2}\right) \approx \nu_1 + \frac{\nu_1}{2}\varepsilon\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

$$\approx \nu_1 + \nu_1\frac{\varepsilon}{2}\frac{\Delta r}{r_1 r_2}$$

$$\frac{\Delta\nu}{\nu} = \frac{\nu_2 - \nu_1}{\nu_1} \approx \frac{\varepsilon}{2}\frac{\Delta r}{r_1^2}$$

når $\Delta r = r_1 - r_2 \ll r_1$