EKSAMEN I TEORETISK FYSIKK III B, 15. mai 1979

LØSNINGSFORSLAG.

Oppgave 1.

$$\mathcal{L}_\psi = i\psi_R^+(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R + i\psi_L^+(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i\psi_R^+\sigma^\mu\partial_\mu\psi_R + i\psi_L^+\tilde{\sigma}^\mu\partial_\mu\psi_L$$

$$(\sigma^\mu) = (I, \vec{\sigma}) \quad (\tilde{\sigma}^\mu) = (I, -\vec{\sigma})$$

ⓐ

Kanonisk konjugerte fireimpulser

$$\Pi^\mu_{\psi_R} = \frac{\partial\mathcal{L}_\psi}{\partial(\partial_\mu\psi_R)} = i\psi_R^+\sigma^\mu \;;\; \Pi^\mu_{\psi_L} = i\psi_L^+\tilde{\sigma}^\mu \;;\; \Pi^\mu_{\psi_R^+} = \Pi^\mu_{\psi_L^+} = 0$$

Euler-Lagrange ligningene

$$\frac{\partial\mathcal{L}}{\partial\psi_R^+} = i\sigma^\mu\partial_\mu\psi_R = 0 \qquad \frac{\partial\mathcal{L}}{\partial\psi_L^+} = i\tilde{\sigma}^\mu\partial_\mu\psi_L = 0$$

ⓑ

Finner først normalmodene, som må ha form $\tilde{\psi}_a(\vec{k})e^{-i\omega x^0 + i\vec{k}\cdot\vec{x}}$ for $a = R, L$

E.L ⇒

$$(\omega - \vec{\sigma}\cdot\vec{k})\tilde{\psi}_R(\vec{k}) = \begin{pmatrix} \omega - k^3 & -k^- \\ -k^+ & \omega + k^3 \end{pmatrix}\tilde{\psi}_R(\vec{k}) = 0 \text{ , der } k^\pm = k^1 \pm i k^2$$

$$(\omega + \vec{\sigma}\cdot\vec{k})\tilde{\psi}_L(\vec{k}) = 0 \quad \Rightarrow \quad \tilde{\psi}_L(\vec{k}) = \tilde{\psi}_R(-\vec{k})$$

Egenverdiligning med egenverdier $\omega_\pm = \pm|\vec{k}| \equiv \pm k^0$, og egenvektorer $\tilde{\psi}_R^+(\vec{k}) \equiv u_R(\vec{k})$, $\tilde{\psi}_R^-(\vec{k}) \equiv v_R(-\vec{k})$ $[\vec{\sigma}\cdot\vec{k}$ hermitisk $\Rightarrow \psi^-(\vec{k})^+\psi^+(\vec{k}) = 0]$

$$\begin{pmatrix} k^0 - k^3 & -k^- \\ -k^+ & k^0 + k^3 \end{pmatrix} u_R(\vec{k}) = 0 \quad \Rightarrow \quad u_R(\vec{k}) = N\begin{pmatrix} k^- \\ k^0 - k^3 \end{pmatrix}$$

Normering

$$u_R^+(\vec{k})u_R(\vec{k}) = |N|^2(k^+k^- + |\vec{k}|^2 + (k^3)^2 - 2|\vec{k}|k^3) = 2|N|^2(1 - \hat{k}^3) = 1 \quad \Rightarrow$$

$$\underline{u_R(\vec{k}) = \frac{e^{i\alpha}}{\sqrt{2(1-\hat{k}^3)}}\begin{pmatrix} \hat{k}^- \\ 1 - \hat{k}^3 \end{pmatrix} = \frac{e^{i\beta}}{\sqrt{2(1+\hat{k}^3)}}\begin{pmatrix} 1 + \hat{k}^3 \\ \hat{k}^+ \end{pmatrix}}$$

Fasen α kan velges vilkårlig. $\left(tg(\alpha - \beta) = \frac{\hat{k}^2}{\hat{k}^1}\right)$ Her er $\hat{k}^i = \frac{k^i}{|\vec{k}|}$

$$\begin{pmatrix} -k^0 - k^3 & -k^- \\ -k^+ & -k^0 + k^3 \end{pmatrix}\tilde{\psi}_R^-(\vec{k}) = 0 \quad \Rightarrow \quad \underline{v_R(\vec{k}) = e^{i\gamma}u_R(\vec{k})}$$

Fasen γ kan velges vilkårlig.

og altså

$$\underline{u_L(\vec{k}) = v_L(\vec{k}) = u_R(-\vec{k})} \qquad \text{(modulo fasen)}$$

Settet $\{\tilde{\psi}_a^\pm(\vec{k})\}$ er en basis for hver $\vec{k}$, ved kompletthet av $\{e^{i\vec{k}\cdot\vec{x}}\}$ kan derfor den generelle løsningen skrives på formen

$$\psi_a(x) = \int\frac{d^3k}{(2\pi)^{3/2}}\, b_a(\vec{k})\tilde{\psi}_a^+(\vec{k})e^{-ik^0x^0 + i\vec{k}\cdot\vec{x}} + d_a^+(-\vec{k})\tilde{\psi}_a^-(\vec{k})e^{ik^0x^0 + i\vec{k}\cdot\vec{x}}$$

$$= \int\frac{d^3k}{(2\pi)^{3/2}}\left[b_a(\vec{k})u_a(\vec{k})e^{-ikx} + d_a^+(\vec{k})v_a(\vec{k})e^{ikx}\right] \quad a = R, L$$

der b_a, d_a er vilkårlige komplekse funksjoner, $kx \equiv k_\mu x^\mu = k^0x^0 - k^ix^i$.

(c) Kanonisk energi-impuls tensor

$$\underline{T_{kan}^{\mu\nu}} = \Pi^{\mu}_{\psi_R}\partial^\nu\psi_R + \Pi^{\mu}_{\psi_L}\partial^\nu\psi_L - \eta^{\mu\nu}\mathcal{L}_\psi = \underline{i\psi_R^+\sigma^\mu\partial^\nu\psi_R + i\psi_L^+\tilde{\sigma}^\mu\partial^\nu\psi_L}$$

fordi $\mathcal{L}_\psi = 0$ ved bevegelsesligh.

$$P_\mu = \int d^3x\ T^0{}_\mu = \int \frac{d^3q}{(2\pi)^{3/2}}\frac{d^3k}{(2\pi)^{3/2}} *$$

$$\sum_{a=R,L} * \{ b_a^+(\vec q)\, k_\mu\, b_a(\vec k)\, u_a^+(\vec q) u_a(\vec k) \int d^3x\ e^{i(\vec k-\vec q)\vec x}\ e^{-i(k^0-q^0)x^0}$$
$$+ d_a(\vec q)\, k_\mu\, b_a(\vec k)\, v_a^+(\vec q) u_a(\vec k) \int d^3x\ e^{i(\vec k+\vec q)\vec x}\ e^{-i(k^0+q^0)x^0}$$
$$- b_a^+(\vec q)\, k_\mu\, d_a^+(\vec k)\, u_a^+(\vec q) v_a(\vec k) \int d^3x\ e^{-i(\vec k+\vec q)\vec x}\ e^{i(k^0+q^0)x^0}$$
$$- d_a(\vec q)\, k_\mu\, d_a^+(\vec k)\, v_a^+(\vec q) v_a(\vec q) \int d^3x\ e^{-i(\vec k-\vec q)x}\ e^{i(k^0-q^0)x^0} \}$$

Integrerer over d^3x og d^3q og bruker $v_a^+(-\vec k)u_a(\vec k) = u_a^+(-\vec k)v_a(\vec k) = 0$

$$\underline{P_\mu = \sum_{a=R,L}\int d^3k\ k_\mu [\, b_a^+(\vec k) b_a(\vec k) - d_a(\vec k) d_a^+(\vec k)]}$$

(d)

$\mathcal{L}_\psi$ er invariant under $U(1)\times U(1)$ indre symmetrigruppe

$$\psi_R \to e^{-i\phi_R}\psi_R \qquad Q_a = \frac{\partial}{\partial\phi_a}\psi_a' \Big|_{\phi_a=0} = -i\psi_a \qquad a=R,L$$
$$\psi_L \to e^{-i\phi_L}\psi_L$$

Konserverte strømmer

$$\underline{j_R^\mu} = \Pi^\mu_{\psi_R} Q_R = \underline{\psi_R^+\sigma^\mu\psi_R} \qquad \underline{j_L^\mu = \psi_L^+\tilde\sigma^\mu\psi_L}$$ (modulo fortegn og normering)

$\boxed{\text{Oppgave 2.}}$

$$\mathcal{L}_\Phi = \partial^\mu\phi^*\partial_\mu\phi + \mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 \ ; \ \mathcal{L}_{\psi\Phi} = -G(\psi_R^+\Phi^*\psi_L + \psi_L^+\Phi\psi_R)$$

(a) For $\psi=0$ blir energitettheten

$$\mathcal{H}[\Phi] = T^{00} = 2\,\partial^0\Phi^*\partial^0\Phi - \mathcal{L} = \underbrace{\partial^0\phi^*\partial^0\Phi + (\nabla\Phi)^*\nabla\Phi}_{\text{Positivt definitt}} + \lambda(\Phi^*\Phi)^2 - \mu^2\Phi^*\Phi$$

Minimum når Φ er konstant og lik

$$\underline{\Phi_0 = \frac{\mu}{\sqrt{2\lambda}}e^{i\alpha}}\ , \text{fasen } \alpha \text{ vilkårlig.} \qquad \mathcal{H}[\Phi_0] = -\frac{\mu^4}{4\lambda}$$

(b)

$$\psi_R \to e^{-i\phi_R}\psi_R\ ,\ \psi_L \to e^{-i\phi_L}\psi_L\ ,\ \Phi \to e^{i\phi_R - i\phi_L}\Phi$$

holder $\mathcal{L}$ invariant.

(c)

Todimensjonal gaugegruppe, så to gaugefelter $A_\mu^{R,L}$.

$$\underline{\mathcal{L}_{gf} = -\frac14\sum_{a=R,L} F^a_{\mu\nu}F^{a\mu\nu}} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \qquad a=R,L$$

Men generelt er

$$\mathcal{L}_{gf} = \frac{1}{4e^2}\sum_\alpha Z_\alpha R^\alpha_{\mu\nu}R^{\alpha\mu\nu} \qquad R^\alpha_{\mu\nu} = [D^\alpha_\mu, D^\alpha_\nu] \qquad \alpha = R, L, \Phi$$

gaugeinvariant og akseptabel (og framtvinges automatisk ved renormalisering av den kvantiserte teorien?)

(d) $D_\mu^R = \partial_\mu - ie_R A_\mu^R \qquad D_\mu^L = \partial_\mu - ie_L A_\mu^L$

$$\psi_R(x) \to e^{-i\alpha_R(x)}\psi_R(x) \qquad \Rightarrow \quad A_\mu^R \to A_\mu^R - \frac{1}{e_R}\partial_\mu \alpha_R$$

$$\psi_L(x) \to e^{-i\alpha_L(x)}\psi_L(x) \qquad \Rightarrow \quad A_\mu^L \to A_\mu^L - \frac{1}{e_R}\partial_\mu \alpha_L$$

$\Phi(x) \to e^{i\alpha_R(x) - i\alpha_L(x)}\Phi(x)$ krever da

$$D_\mu^\Phi = \partial_\mu - ie_L A_\mu^L + ie_R A_\mu^R$$

(e) Gitt $\Phi = \|\Phi\| e^{i\,\mathrm{Arg}(\Phi(x))}$ og gjør en gaugetransf. som i (d) med $\alpha_L(x) - \alpha_R(x) = \mathrm{Arg}[\Phi(x)]$. Etter denne er

$$\underline{\mathcal{L}_\Phi(\Phi, D_\mu\Phi)} = (D_\mu\Phi)^*(D^\mu\Phi) - V[\|\Phi\|]$$
$$= \underline{\partial_\mu\|\Phi\|\,\partial^\mu\|\Phi\| + \|\Phi\|^2(e_R^2+e_L^2)C_\mu C^\mu + \mu^2\|\Phi\|^2 - \lambda\|\Phi\|^4}$$

med

$$C_\mu = \frac{e_R}{\sqrt{e_R^2+e_L^2}}A_\mu^R - \frac{e_L}{\sqrt{e_R^2+e_L^2}}A_\mu^L$$

(f) Til 2^{nd} orden i feltene

$$\underline{\mathcal{L}_\psi} = i\psi_R^+\sigma^\mu\partial_\mu\psi_R + i\psi_L^+\tilde{\sigma}^\mu\partial_\mu\psi_L = \underline{i\bar{\Psi}\gamma^\mu\partial_\mu\Psi}$$

$$\underline{\mathcal{L}_{\Phi\psi}} = -G\|\Phi_0\|(\psi_R^+\psi_L + \psi_L^+\psi_R) = \underline{-\frac{\mu G}{\sqrt{2\lambda}}\bar{\Psi}\Psi}$$

$$\underline{\mathcal{L}_\Phi} = \frac{\mu^2}{4\lambda} + \partial_\mu\Phi_1\partial^\mu\Phi_1 - 2\mu^2\Phi_1^2 + \frac{\mu^2}{2\lambda}(e_R^2+e_L^2)C_\mu C^\mu$$

$$\underline{\mathcal{L}_{gf}} = -\frac{1}{4}F_{\mu\nu}^R F^{R\mu\nu} - \frac{1}{4}F_{\mu\nu}^L F^{L\mu\nu}$$
$$= \underline{-\frac{1}{4}F_{\mu\nu}^B F^{B\mu\nu} - \frac{1}{4}F_{\mu\nu}^C F^{C\mu\nu}}$$

der

$$F_{\mu\nu}^B = \partial_\mu B_\nu - \partial_\nu B_\mu \qquad F_{\mu\nu}^C = \partial_\mu C_\nu - \partial_\nu C_\mu$$

$$\begin{pmatrix} B_\mu \\ C_\mu \end{pmatrix} = \begin{pmatrix} \frac{e_R}{\sqrt{e_R^2+e_L^2}} & \frac{e_L}{\sqrt{e_R^2+e_L^2}} \\ \frac{-e_L}{\sqrt{e_R^2+e_L^2}} & \frac{e_R}{\sqrt{e_R^2+e_L^2}} \end{pmatrix} \begin{pmatrix} A_\mu^L \\ A_\mu^R \end{pmatrix}$$

(Siden denne er ortogonal bevares den kvadratiske formen $F_{\mu\nu}^a F^{a\mu\nu}$,

Dette har den ønskede form, med

$$\underline{m = \frac{\mu G}{\sqrt{2\lambda}}} \qquad \underline{\varkappa^2 = 2\mu^2} \qquad \underline{M^2 = \frac{\mu^2(e_R^2+e_L^2)}{\lambda}}$$