Eksamen i fag 71567 IKKE-REGULÆRE APPROXIMASJONS-METODER I FYSIKKEN.

19.12.1985 $09^{\underline{00}} - 15^{\underline{00}}$.

Løsningsforslag.

Oppgave 1.

$$x f''' + 3f'' + 2f = 0$$

a) Regulært singulært punkt i $x = 0$.
Irregulært singulært punkt i $x = \infty$.

b1) $x \to \infty$:

$$f = e^{S_0 + S_1 + \cdots} \Rightarrow$$

$$x S_0'^3 + 2 = 0 \Rightarrow S_0' = \omega\left(\frac{2}{x}\right)^{1/3} \;;\; \omega = -1, e^{\pm i\pi/3}$$

$$3x S_0'^2 S_1' + 3 S_0'^2 + 3x S_0' S_0'' + \underbrace{S_0'' + x S_0''}_{\text{neglisjerbare}} = 0 \Rightarrow$$

$$S_1' + \frac{1}{x} + \frac{S_0''}{S_0'} = 0 \Rightarrow S_1' = -\frac{2}{3x}$$

Altså

$$S = 3\omega\left(\frac{x}{2}\right)^{2/3} - \frac{2}{3}\ln x + \cdots$$

$$\boxed{f(x) \sim \text{konst}\cdot\left(\frac{2}{x}\right)^{2/3} e^{3\omega(x/2)^{2/3}} \;;\; \omega = -1, e^{\pm i\pi/3}}$$

b2) $x \to 0$:

$$\boxed{f(x) \sim x^{\nu} \Rightarrow \text{Indicielign. } (\nu-1)\nu(\nu+1) = 0 \Rightarrow \nu = -1, 0, 1}$$

Oppgave 2.

$$F(x) = \int_0^\infty dt\, e^{-xt-t^{-2}}$$

a) $x \to \infty$

Eksponenten har et (bevegelig) maksimum ved $x - 2t_0^{-3} = 0$, dvs. $t_0 = (x/2)^{-1/3}$

$$\frac{d^2}{dt^2}\left(xt + t^{-2}\right)\Big|_{t=t_0} = 6t_0^{-4} = 6\,(x/2)^{4/3}$$

Innfører ny variabel

$$t = (x/2)^{-1/3} + \frac{1}{\sqrt{6}}\left(\frac{x}{2}\right)^{-2/3} u$$

og tilnærmer

$$F(x) \simeq \frac{1}{\sqrt{6}}\left(\frac{x}{2}\right)^{-2/3} e^{-3(x/2)^{2/3}} \underbrace{\int_{-\infty}^{\infty} du\, e^{-\frac{1}{2}u^2}}_{\sqrt{2\pi}}$$

$$\boxed{F(x) \simeq \sqrt{\frac{\pi}{3}}\left(\frac{x}{2}\right)^{-2/3} e^{-3(x/2)^{2/3}}}$$

b) $x \to 0^+$

$$F''(x) = \int_0^\infty dt\, t^2 e^{-xt-t^{-2}} = \int_0^\infty dt\, t^2 e^{-xt}\left(1 - \frac{1}{t^2} + \cdots\right)$$

$$= \frac{2}{x^3} - \frac{1}{x} + \cdots$$

Integrerer opp

$$F(x) = \frac{1}{x} - x\ln x + \text{konst.} + \cdots$$

$$\text{konst.} = \lim_{x\to 0^+} \int_0^\infty dt\, e^{-xt}\left(e^{-t^{-2}} - 1\right) = \int_0^\infty dt \left(e^{-t^{-2}} - 1\right) = -\sqrt{\pi}$$

$$\boxed{F(x) \simeq \frac{1}{x} - \sqrt{\pi} - x\ln x + \cdots}$$

Oppgave 3.

a) Grenseskikt ved $x=0$ av tykkelse $\delta = \varepsilon^{\frac{1}{n+1}}$.

b) Ytre løsninger:

$$y_+(x) = b\, e^{-\frac{1}{2}(x^2-1)} \qquad x>0$$
$$y_-(x) = a\, e^{-\frac{1}{2}(x^2-1)} \qquad x<0$$

Indre løsning: $x = \delta X,\ Y(X) = y(x) \Rightarrow Y'' + X^n Y' = 0 \Rightarrow$

$$Y(X) = C_1 + C_2 \int_{-\infty}^{X} dX'\, e^{-\frac{1}{n+1}X'^{\,n+1}}$$

Matching:

$$C_1 = a\sqrt{e} \qquad C_2 = (b-a)\sqrt{e}\cdot N$$

der

$$N^{-1} = \int_{\infty}^{\infty} dX'\, e^{-\frac{1}{n+1}X'^{\,n+1}} = 2\cdot\left(\frac{1}{n+1}\right)^{\frac{n}{n+1}} \Gamma\left(\frac{1}{n+1}\right).$$

Uniform løsning:

$$y_{unif}(x) = a\, e^{-\frac{1}{2}(x^2-1)} + (b-a)\, e^{-\frac{1}{2}(x^2-1)}\left[N\int_{-\infty}^{X} dX'\, e^{-\frac{1}{n+1}X'^{\,n+1}}\right].$$

Oppgave 4.

a) $Q(x_\pm;\lambda)=0$. To klassiske vendepunkter. Uendelig intervall. Glatthetsbetingelser på Q.

b) $$\frac{1}{\varepsilon}\int_{x_-}^{x_+} dx\,\sqrt{\lambda-\omega^2x^2} = \frac{\lambda}{\omega\varepsilon}\underbrace{\int_{-1}^{1} dt\,\sqrt{1-t^2}}_{\pi/2} = (n+\tfrac{1}{2})\pi$$

$$\Rightarrow \boxed{\lambda_n = (2n+1)\varepsilon\omega}$$ Eksakte egenverdier.

c) $$-y''+\left[\frac{\ell(\ell+1)}{r^2}+V(r)\right]y(r)=\lambda y(r) \qquad ; r\geq 0$$

- Skriver $r=e^x$, $-\infty<x<\infty \Rightarrow$

$$+e^{-2x}\left\{-\frac{d^2}{dx^2}+\frac{d}{dx}+\ell(\ell+1)+e^{2x}\left(V(e^x)-\lambda\right)\right\}y(x)=0$$

- Transformerer bort $\frac{d}{dx}$-leddet ved å skrive

$$y(x)=e^{\frac{1}{2}x}Y(x) \Rightarrow$$

$$\left[-\frac{d^2}{dx^2}+\ell(\ell+1)+\frac{1}{4}+e^{2x}\left(V(e^x)-\lambda\right)\right]Y(x)=0$$

- WKB-kvantiseringsbetingelsen blir

$$\int_{x_-}^{x_+} dx\; e^x\sqrt{\lambda-(\ell+\tfrac{1}{2})^2e^{-2x}-V(e^x)} = (n+\tfrac{1}{2})\pi$$

- Transformerer tilbake, $r=e^x$, og får

$$\boxed{\int_{r_-}^{r_+} dr\,\sqrt{\lambda-\frac{(\ell+\frac{1}{2})^2}{r^2}-V(r)} = (n+\tfrac{1}{2})\pi}$$

d) Harmonisk oscillator

$$\int_{r_-}^{r_+} \frac{dr}{r} \sqrt{-\omega^2 r^4 + \lambda r^2 - (\ell+\tfrac{1}{2})^2} = (n+\tfrac{1}{2})\pi$$

$r^2 = \varrho \Rightarrow$

$$\int_{\varrho_-}^{\varrho_+} \frac{d\varrho}{\varrho} \sqrt{-\frac{\omega^2}{4}\varrho^2 + \frac{\lambda}{4}\varrho - \frac{(\ell+\frac{1}{2})^2}{4}} = (n+\tfrac{1}{2})\pi$$

Hydrogenatom:

$$\int_{r_-}^{r_+} \frac{dr}{r} \sqrt{\lambda r^2 + e^2 r - (\ell+\tfrac{1}{2})^2} = (n+\tfrac{1}{2})\pi$$

Samme kvantiserings-betingelse

$$F(\alpha,\beta,\gamma) = (n+\tfrac{1}{2})\pi$$

der

	Harm. oscillator	Hydrogen atom
α	$\omega^2/4$	$-\lambda$
β	$\lambda/4$	e^2
γ	$(\ell+\frac{1}{2})^2/4$	$(\ell+\frac{1}{2})^2$

e) Beregner $F(\alpha,\beta,\gamma) = \int_{\varrho_-}^{\varrho_+} \frac{d\varrho}{\varrho}\sqrt{-\alpha\varrho^2 + \beta\varrho - \gamma}$

$$= \sqrt{\alpha}\int_{x_-}^{x_+} \frac{dx}{x}\sqrt{(x_+-x)(x-x_-)} \qquad x_+ + x_- = \beta/\alpha,\quad x_+ x_- = \gamma/\alpha$$

$$= \frac{\pi}{2}\sqrt{\alpha}\left[\frac{\beta}{\alpha} - 2\sqrt{\frac{\gamma}{\alpha}}\right] = \frac{\pi}{2}\left(\frac{\beta}{\sqrt{\alpha}} - 2\sqrt{\gamma}\right)$$

Harmonisk oscillator:

$$\frac{\pi}{2}\left[\frac{\lambda}{2\omega} - (\ell+\tfrac{1}{2})\right] = (n+\tfrac{1}{2})\pi \quad \therefore \quad \boxed{\lambda_n = (4n+2\ell+3)\omega}$$

Hydrogenatom:

$$\frac{\pi}{2}\left[\frac{e^2}{\sqrt{-\lambda}} - (2\ell+1)\right] = (n+\tfrac{1}{2})\pi \quad \therefore \quad \boxed{\lambda_n = -\frac{e^4}{4(n+\ell+1)^2}}$$

(Begge kvantiseringsformlene er eksakte.)