Klassisk feltmekanikk   Eks. 12.12.89

Løsninger.

I a) Noethers teorem:

Har et system en kontinuerlig symmetri, så har det en tilhørende bevarelsessetning.

Systemet har en symmetri hvis det finnes en transformasjon $\varphi_\alpha(x) \to \varphi_\alpha'(x)$ som, eventuelt kombinert med en koordinattransformasjon $x^\mu \to x^{\mu'}$, fører løsninger $\varphi_\alpha(x)$ for systemet over i nye løsninger $\varphi_\alpha'(x')$.

b/α) Rotasjon om x-aksen:

$$\frac{\partial \varphi^{2\prime}}{\partial x^\mu} = \frac{\partial \varphi^2}{\partial x^\mu}\cos\theta - \frac{\partial \varphi^3}{\partial x^\mu}\sin\theta = \frac{\partial \varphi_2'}{\partial x^\mu}$$

$$\frac{\partial \varphi^{3\prime}}{\partial x^\mu} = \frac{\partial \varphi^2}{\partial x^\mu}\sin\theta + \frac{\partial \varphi^3}{\partial x^\mu}\cos\theta = \frac{\partial \varphi_3'}{\partial x^\mu}$$

Det nye feltets verdi på et sted er bestemt av det gamle feltets verdi på samme sted: Ingen koord. forandr. $x^{\mu\prime} = x^\mu$

$$\underline{\mathcal{L}(\varphi'(x))} = \frac{1}{2}K\sum_\mu\left(\frac{\partial \varphi^{2\prime}}{\partial x^\mu}\frac{\partial \varphi_2'}{\partial x_\mu} + \frac{\partial \varphi^{3\prime}}{\partial x^\mu}\frac{\partial \varphi_3'}{\partial x_\mu}\right)$$

$$= \frac{1}{2}K\sum_\mu\left\{\frac{\partial \varphi^2}{\partial x^\mu}\frac{\partial \varphi_2}{\partial x_\mu}(\cos^2\theta + \sin^2\theta) + 2\frac{\partial \varphi^2}{\partial x^\mu}\frac{\partial \varphi_3}{\partial x_\mu}(\cos\theta\sin\theta - \sin\theta\cos\theta) + \frac{\partial \varphi^3}{\partial x^\mu}\frac{\partial \varphi_3}{\partial x_\mu}(\sin^2\theta + \cos^2\theta)\right\}$$

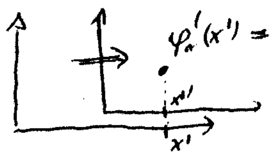
$$= \frac{1}{2}K\sum_\mu\left(\frac{\partial \varphi^2}{\partial x^\mu}\frac{\partial \varphi_2}{\partial x_\mu} + \frac{\partial \varphi^3}{\partial x^\mu}\frac{\partial \varphi_3}{\partial x_\mu}\right) = \underline{\mathcal{L}(\varphi(x))}$$

Altså strengen er <u>symmetrisk ved rotasjonen</u> da Lagrange-tettheten er invariant.

β) Lorentz-transformasjon langs x-aksen.

$$\frac{\partial}{\partial x^0} = \frac{\partial x^{\nu\prime}}{\partial x^0}\frac{\partial}{\partial x^{\nu\prime}} = \cosh\chi\frac{\partial}{\partial x^{0\prime}} - \sinh\chi\frac{\partial}{\partial x^{1\prime}}$$

$$\frac{\partial}{\partial x^1} = \frac{\partial x^{\nu\prime}}{\partial x^1}\frac{\partial}{\partial x^{\nu\prime}} = -\sinh\chi\frac{\partial}{\partial x^{0\prime}} + \cosh\chi\frac{\partial}{\partial x^{1\prime}}$$



$$\varphi_\alpha'(x') = \varphi_\alpha(x)$$

$$\mathcal{L}(\varphi(x)) = \frac{1}{2}K\sum_\alpha\left(\frac{\partial \varphi^\alpha}{\partial x^0}\frac{\partial \varphi_\alpha}{\partial x^0} - \frac{\partial \varphi^\alpha}{\partial x^1}\frac{\partial \varphi_\alpha}{\partial x^1}\right)$$

$$= \frac{1}{2}K\sum_\alpha\left\{\frac{\partial \varphi^{\alpha\prime}}{\partial x^{0\prime}}\frac{\partial \varphi_\alpha'}{\partial x^{0\prime}}(\cosh^2\chi - \sinh^2\chi) - 2\frac{\partial \varphi_\alpha'}{\partial x^{0\prime}}\frac{\partial \varphi_\alpha'}{\partial x^{1\prime}}(\cosh\chi\sinh\chi - \cosh\chi\sinh\chi) + \frac{\partial \varphi^{\alpha\prime}}{\partial x^{1\prime}}\frac{\partial \varphi_\alpha'}{\partial x^{1\prime}}(\sinh^2\chi - \cosh^2\chi)\right\}$$

$$= \frac{1}{2}K\sum_\alpha\left(\frac{\partial \varphi^{\alpha\prime}}{\partial x^{0\prime}}\frac{\partial \varphi_\alpha'}{\partial x^{0\prime}} - \frac{\partial \varphi_\alpha'}{\partial x^{1\prime}}\frac{\partial \varphi_\alpha'}{\partial x^{1\prime}}\right) = \mathcal{L}(\varphi'(x'))$$

$\int \mathcal{L}(\varphi(x))\,dx^0dx^1 = \int \mathcal{L}(\varphi'(x'))\,dx^{0\prime}dx^{1\prime}$, altså <u>symmetrisk ved Lorentz-transformasjonen.</u>

c α) Bevart strømtetthet: $J^\mu = \pi^\mu{}_\alpha Q^\alpha - R^\mu \mathcal{L}$

Her: $Q^2 = \left.\frac{d\varphi^{2\prime}}{d\theta}\right|_{\theta=0} = (-\varphi^2\sin\theta - \varphi^3\cos\theta)_{\theta=0} = -\varphi^3$ $x^{\mu\prime} = x^\mu$ $R^\mu = \left.\frac{dx^{\mu\prime}}{d\theta}\right|_{\theta_0} = 0$

$Q^3 = \left.\frac{d\varphi^{3\prime}}{d\theta}\right|_{\theta=0} = (\varphi^2\cos\theta - \varphi^3\sin\theta)_{\theta=0} = \varphi^2$ $\pi^\mu{}_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)} = K\partial^\mu\varphi_\alpha$

$$J^0 = \pi^0{}_\alpha Q^\alpha = K\left(\partial^0\varphi_2(-\varphi^3) + \partial^0\varphi_3\cdot\varphi^2\right) = \frac{K}{c}\left(\varphi^2\frac{\partial \varphi^3}{\partial t} - \varphi^3\frac{\partial \varphi^2}{\partial t}\right)$$

$$J^1 = \pi^1{}_\alpha Q^\alpha = -K\left(\varphi^2\frac{\partial \varphi^3}{\partial x^1} - \varphi^3\frac{\partial \varphi^2}{\partial x^1}\right)$$

Lokal bevarelsessetning: $\underline{\frac{\partial}{\partial t}\left(\frac{1}{c}J^0\right) + \frac{\partial}{\partial x}J^1 = 0} \Rightarrow K\left\{\varphi^2\left(\frac{1}{c^2}\frac{\partial^2\varphi^3}{\partial t^2} - \frac{\partial^2\varphi^3}{\partial x^2}\right) - \varphi^3\left(\frac{1}{c^2}\frac{\partial^2\varphi^2}{\partial t^2} - \frac{\partial^2\varphi^2}{\partial x^2}\right)\right\} = 0$

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c) forts.

Globalt: (Strengen går fra x_a til x_b)

$$\int \frac{\partial}{\partial t} \frac{1}{c} j^0(x,t)\, dx = -\int_{x_a}^{x_b} \frac{\partial}{\partial x} j^1\, dx = -\left[j^1(x_a) - j^1(x_b) \right] = 0 \quad \text{da } \varphi^a(x_a) = 0 = \varphi^a(x_b)$$

$$\frac{d}{dt} \int \frac{K}{c^2}\left(\varphi^2 \frac{\partial \varphi^3}{\partial t} - \varphi^3 \frac{\partial \varphi^2}{\partial t}\right) dx = \frac{K}{\rho c^2} \frac{d L_x}{dt} = 0$$

med dreieimpuls om x-aksen

$$L_x = \int \rho \left(\varphi^2 \frac{\partial \varphi^3}{\partial t} - \varphi^3 \frac{\partial \varphi^2}{\partial t}\right) dx = \text{konst.} \qquad \rho = \text{masse pr lengdeenhet}$$

Med betydningen av φ^2 og φ^3

$$\underline{L_x = \int \left(y(x)\, p_z(x) - z(x)\, p_y(x)\right) dx} \qquad p_y = \rho \dot{y}, \quad p_z = \rho \dot{z}.$$

β) Her

$$Q^\alpha = \frac{d\varphi^{\alpha\prime}(x')}{d\chi}\Big|_{\chi=0} = \frac{\partial \varphi^{\alpha\prime}}{\partial x^{\mu\prime}} \frac{dx^{\mu\prime}}{d\chi}\Big|_{\chi=0} = \frac{\partial \varphi^\alpha}{\partial x^0}(-x^1) + \frac{\partial \varphi^\alpha}{\partial x^1}(-x^0)$$

$$R^0 = \frac{dx^{0\prime}}{d\chi}\Big|_{\chi=0} = \left(x^0 \sinh\chi - x^1 \cosh\chi\right)\Big|_{\chi=0} = -x^1 \qquad \pi^\mu_\alpha = K \partial^\mu \varphi_\alpha$$

$$R^1 = \frac{dx^{1\prime}}{d\chi}\Big|_{\chi=0} = -x^0$$

$\frac{\partial}{\partial t}\left(\frac{1}{c} j^0\right) + \frac{\partial}{\partial x^1} j^1 = 0$ med:

$$j^0 = \pi^0_\alpha Q^\alpha - R^0 \mathcal{L} = \pi^0_\alpha\, \partial_0 \varphi^\alpha (-x^1) + \pi^0_\alpha\, \partial_1 \varphi^\alpha (-x^0) - (-x^1)\mathcal{L} =$$

$$= -x^1\left(\pi^0_\alpha \partial^0 \varphi^\alpha - \mathcal{L}\right) + x^0 \pi^0_\alpha \partial^1 \varphi^\alpha = -\left(x^1 T^{00} - x^0 T^{01}\right)$$

$$= K \partial^0 \varphi_\alpha\, \partial_0 \varphi^\alpha (-x^1) + K \partial^0 \varphi_\alpha\, \partial_1 \varphi^\alpha (-x^0) - (-x^1) \frac{K}{2}\left(\partial_0 \varphi^\alpha \partial^0 \varphi_\alpha + \partial_1 \varphi^\alpha \partial^1 \varphi_\alpha\right)$$

$$= -x^1 \frac{K}{2}\left(\partial^0 \varphi_\alpha \partial_0^0 \varphi^\alpha + \partial^1 \varphi_\alpha \partial^1 \varphi^\alpha\right) + x^0 K \partial^0 \varphi_\alpha \partial^1 \varphi^\alpha$$

$$j^1 = \pi^1_\alpha Q^\alpha - R^1 \mathcal{L} = \pi^1_\alpha \partial_0 \varphi^\alpha (-x^1) + \pi^1_\alpha \partial_1 \varphi^\alpha (-x^0) - (-x^0)\frac{K}{2}\left(\partial_0 \varphi^\alpha \partial^0 \varphi_\alpha + \partial_1 \varphi^\alpha \partial^1 \varphi_\alpha\right)$$

$$= -x^1 \pi^1_\alpha \partial^0 \varphi^\alpha + x^0\left(\pi^1_\alpha \partial^1 \varphi^\alpha - g^{11} \mathcal{L}\right) = -\left(x^1 T^{10} - x^0 T^{11}\right) \qquad (g^{11} = -1)$$

$$= -x^1 K \partial^1 \varphi_\alpha \partial^0 \varphi^\alpha + x^0\left(\frac{K}{2}\left(\partial^0 \varphi_\alpha \partial^0 \varphi^\alpha + \partial^1 \varphi_\alpha \partial^1 \varphi^\alpha\right)\right)$$

Globalt bevart størrelse:

$$\frac{d}{dt} \int \frac{1}{c} j^0(x)\, dx = 0$$

$$-\int\left(x^1 T^{00} - x^0 T^{01}\right) dx = -\int\left(x^1 \mathcal{H} - ct\, c g^1\right) dx = -\left(E R - G^1 c^2 t\right) = \text{konst.}$$

"Tyngdepunkt bevarelse": Energi tyngdepunktet går med konstant hastighet.

$$\underline{R = R_0 + \frac{G c^2}{E}(t - t_0)}$$

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II a) $g_{\mu\nu} = G_{\mu\nu} + h_{\mu\nu} = g_{\nu\mu}$ $\qquad \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial h_{\mu\nu}}{\partial x^\lambda}$

$$\Gamma_{k0l} + \Gamma_{l0k} = \frac12\left(\frac{\partial g_{k0}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^0} - \frac{\partial g_{0l}}{\partial x^k}\right) + \frac12\left(\frac{\partial g_{l0}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^0} - \frac{\partial g_{0k}}{\partial x^l}\right) = \frac{\partial g_{kl}}{\partial x^0} = \frac1c\frac{\partial h_{kl}}{\partial t} = \underline{0}$$

Venstre side:

$$\frac{d^2x^k}{dt^2} = -\Gamma^k_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = -\Gamma^k_{00}c^2 - 2\Gamma^k_{i0}c\frac{dv^i}{dt} - \overbrace{\Gamma^k_{ij}\frac{dv^i}{dt}\frac{dv^j}{dt}}^{\propto \mathcal{O}(v^2)}$$

Høyre side:

$$f^k + 2\varepsilon^k{}_{ij}\frac{dx^i}{dt}\omega^j = -c^2\Gamma^k_{00} + 2\varepsilon^k{}_{ij}\frac{dx^i}{dt}\frac{c}{2}\varepsilon^{jlm}\Gamma_{l0m}$$

$$= -\Gamma^k_{00}c^2 - \frac{dx^i}{dt}c\,g^{kp}(\delta_p^l\delta_i^m - \delta_p^m\delta_i^l)\Gamma_{l0m}$$

$$= -\Gamma^k_{00}c^2 - \frac{dx^i}{dt}c\,g^{kp}(\Gamma_{p0i} - \Gamma_{i0p})$$

$$= \underline{-\Gamma^k_{00}c^2 - 2\Gamma^k_{0i}c\frac{dv^i}{dt}} \qquad (\text{Fra a) } \Gamma_{i0p} = -\Gamma_{p0i})$$

$$f^i = -\Gamma^i_{00}c^2 = -\frac{c^2}{2}g^{ik}\left(\frac{\partial h_{k0}}{\partial x^0} + \frac{\partial h_{k0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^k}\right) = \underline{c^2\left(\frac{\partial h_{i0}}{\partial x^0} - \frac12\frac{\partial h_{00}}{\partial x^i}\right)} \quad \text{da } g^{ik} = G^{ik} + \mathcal{O}(h) = -\delta^{ik}$$

$$\omega^i = \frac c2\varepsilon^{ilm}\Gamma_{l0m} = \frac c4\varepsilon^{ilm}\left(\frac{\partial h_{l0}}{\partial x^m} + \overset{=0}{\frac{\partial h_{lm}}{\partial x^0}} - \frac{\partial h_{0m}}{\partial x^l}\right) = \underline{\frac c2\varepsilon^{ilm}\frac{\partial h_{l0}}{\partial x^m}} \quad \text{idet } \varepsilon^{ilm}\frac{\partial h_{0m}}{\partial x^l} = -\varepsilon^{ilm}\frac{\partial h_{l0}}{\partial x^m}$$

b)

$$R_{00} = \frac{\partial\Gamma^k_{00}}{\partial x^k} - \frac{\partial\Gamma^k_{0k}}{\partial x^0} + \mathcal{O}(h^2) = \underline{\frac{\partial\Gamma^k_{00}}{\partial x^k}} \quad \text{idet } \Gamma^k_{0k} = \frac12 g^{k\lambda}\left(\frac{\partial h_{\lambda0}}{\partial x^k} + \frac{\partial h_{\lambda k}}{\partial x^0} - \frac{\partial h_{0k}}{\partial x^\lambda}\right) = -\frac12\delta^{k\lambda}\frac{\partial h_{\lambda\lambda}}{\partial x^0} = 0$$

$$\text{og } \Gamma^0_{00} = \frac12 g^{0\lambda}\left(\frac{\partial h_{\lambda0}}{\partial x^0} + \frac{\partial h_{\lambda0}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\lambda}\right) = -\frac12\delta^{0\lambda}\left(2\frac{\partial h_{00}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^0}\right) = 0$$

$$R_{0p} = \frac{\partial\Gamma^k_{0p}}{\partial x^k} - \frac{\partial\Gamma^k_{0k}}{\partial x^p} = \frac{\partial\Gamma^k_{0p}}{\partial x^k} = \underline{\frac{\partial\Gamma^k_{0p}}{\partial x^k}} \quad \text{idet } \frac{\partial\Gamma^0_{0p}}{\partial x^0} = \frac12 g^{0\lambda}\left(\frac{\partial^2 h_{\lambda0}}{\partial x^0\partial x^p} + \frac{\partial^2 h_{\lambda p}}{\partial x^0\partial x^0} - \frac{\partial^2 h_{0p}}{\partial x^0\partial x^0}\right) = 0 \ (\text{da } \lambda = 0)$$

$$R_{00} = \frac{\partial\Gamma^k_{00}}{\partial x^k} = -\frac{1}{c^2}\frac{\partial}{\partial x^k}f^k = \frac{8\pi k}{c^4}\left(T_{00} - \frac12 g_{00}T\right) = \frac{4\pi k}{c^4}T^{00} = \frac{4\pi k}{c^4}\rho c^2$$

$$\underline{\nabla\cdot\vec f = \partial_k f^k = -4\pi k\rho}$$

$$\nabla\times\vec\omega = \varepsilon^{ij}{}_k\partial_j\omega^k = g^{ip}g^{jq}\varepsilon_{pqk}\partial_j\frac c2\varepsilon^{klm}\Gamma_{l0m} = -\frac c2 g^{ip}g^{jq}(\delta_p^l\delta_q^m - \delta_p^m\delta_q^l)\partial_j\Gamma_{l0m}$$

$$= -\frac c2 g^{ip}g^{jq}\partial_j(\Gamma_{p0q} - \Gamma_{q0p}) = c\,g^{ip}\partial_j\Gamma^j_{0p} = c\,g^{ip}R_{0p}$$

$$= c\,g^{ip}\frac{8\pi k}{c^4}\left(T_{0p} - \frac12 g_{0p}T\right) = \frac{8\pi k}{c^3}T^{0i} = \underline{\frac{8\pi k}{c^2}\rho\vec V}$$

c) Stavens massetetthet $\rho = \mu\delta^2(r)$ $\quad \mu$ = masse pr lengdeenhet $\quad$ i ro $\vec V = 0$

Totalmasse av en lengde l $\quad \int\rho\, r\,dr\,d\varphi\,dz = \mu l$

$\nabla\times\vec\omega = 0 \Rightarrow \underline{\vec\omega = \nabla\Phi = 0}$ p.g.a. symmetrien

$\nabla\cdot\vec f = -4\pi k\mu\delta^2(r) \Rightarrow \frac1r\frac{\partial}{\partial r}(rf_r) = -4\pi k\mu\delta^2(r)$ da komponentene $\underline{f_\varphi = 0 \ f_z = 0}$ p.g.a. sym.

$\int_0^l dz\int_0^{2\pi}d\varphi\int_0^r \nabla\cdot\vec f\, r\,dr = 2\pi l\,r f_r = -4\pi k\mu l \qquad \underline{f_r = -\frac{2\mu k}{r}}$

$$\underline{\frac{d^2\vec r}{dt^2} = -\frac{2k\mu}{r}\hat e_r}$$