Klassisk felt teori   Kont. 24. 8.90

Løsninger.

1.a Varierer feltet $\delta\psi_\alpha$ med faste grenseverdier:

$$\delta\int \mathcal{L}(\psi_\alpha, \partial_\mu\psi_\alpha, x^\mu)\, d^4x = \int\left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha}\delta\psi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\delta(\partial_\mu\psi_\alpha)\right)dx^4 \quad , \quad \delta(\partial_\mu\psi_\alpha) = \partial_\mu(\delta\psi_\alpha)$$

$$= \int\left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha} - \frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\right)\delta\psi_\alpha\, dx^4 + \int_{\text{grenseflate}}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\delta\psi_\alpha\right)d^3A_\mu$$ etter partiell integrasjon og bruk av gauss sats.

$$= \int\left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha} - \frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\right)\delta\psi_\alpha\, d^4x = 0$$ for alle $\delta\psi_\alpha$ med $\delta\psi_\alpha = 0$ på grenseflatene.

$\Rightarrow$ Felt likningene: $\underline{\dfrac{\partial\mathcal{L}}{\partial\psi_\alpha} - \dfrac{d}{dx^\mu}\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)} = 0}$ for hver α. (Euler-Lagrange likn.)

b/ $\mathcal{L} = \frac{1}{2}K\dfrac{\partial\phi^\alpha}{\partial x^\mu}\dfrac{\partial\phi_\alpha}{\partial x_\mu}$

$$\frac{\partial\mathcal{L}}{\partial\phi_\alpha} = 0 \ , \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)} = K\frac{\partial\phi^\alpha}{\partial x_\mu} \Rightarrow \frac{\partial\mathcal{L}}{\partial\phi_\alpha} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)} = -\frac{\partial}{\partial x^\mu}K\frac{\partial\phi^\alpha}{\partial x_\mu} = -K\frac{\partial^2\phi^\alpha}{\partial x^\mu\partial x_\mu} = \underline{-K\left(\frac{1}{c^2}\frac{\partial^2\phi^\alpha}{\partial t^2} - \frac{\partial^2\phi^\alpha}{\partial x^2}\right) = 0}$$

Bølgelikning, c = fasehastigheten.

c/ Har et system en kontinuerlig symmetri, så finnes det en assosiert bevarelseslov.

d) Viser at hvis $\phi_\alpha(x)$ oppfyller feltlikningen så vil $\phi'_\alpha(x)$ også gjøre det:

$$\frac{1}{c^2}\frac{\partial^2\phi'^\alpha(x)}{\partial t^2} - \frac{\partial^2\phi'^\alpha(x)}{\partial x^2} = \frac{1}{2}\frac{\partial^2\phi^\alpha(x-b)}{\partial t^2} - \frac{\partial^2\phi^\alpha(x-b)}{\partial x^2} = \underline{\frac{1}{c^2}\frac{\partial^2\phi^\alpha(x')}{\partial t'^2} - \frac{\partial^2\phi^\alpha(x')}{\partial x'^2} = 0}\ , \quad \frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = \frac{\partial}{\partial x'}$$

Bevarelseslover. En for hver translasjonsretning:

$$\partial_\mu J^\mu = 0 \qquad J^\mu = \pi^{\alpha\mu}Q_\alpha - R^\mu\mathcal{L} \ , \qquad \pi^{\alpha\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_\alpha)} = K\frac{\partial\phi^\alpha}{\partial x_\mu}$$

Sym. parametre: $\lambda^\kappa = b^\kappa$: $\kappa = 0, 1, \dots$

$$Q_{\alpha\kappa} = \left(\frac{\partial F_\alpha}{\partial\lambda}\right)_{\lambda=0} = \left(\frac{\partial\phi'_\alpha(x^\mu - b^\mu)}{\partial b^\kappa}\right)_{b^\kappa=0} = \underline{-\frac{\partial\phi_\alpha}{\partial x^\kappa}}$$

$$R^\mu_\kappa = \left(\frac{\partial f^\mu}{\partial\lambda}\right)_{\lambda=0} = \left(\frac{\partial x^{\mu\prime}}{\partial b^\kappa}\right)_{b^\kappa=0} = \left(\frac{\partial(x^\mu - b^\mu)}{\partial b^\kappa}\right)_{b^\kappa=0} = -\frac{\partial x^\mu}{\partial x^\kappa} = \underline{-\delta^\mu_\kappa}$$

$$J^\mu_\kappa = \pi^{\alpha\mu}Q_{\alpha\kappa} - R^\mu_\kappa\mathcal{L} = K\frac{\partial\phi^\alpha}{\partial x_\mu}\left(-\frac{\partial\phi_\alpha}{\partial x^\kappa}\right) - (-\delta^\mu_\kappa)\frac{1}{2}K\frac{\partial\phi^\alpha}{\partial x^\nu}\frac{\partial\phi_\alpha}{\partial x_\nu}$$

$$= -K\left(\frac{\partial\phi^\alpha}{\partial x_\mu}\frac{\partial\phi_\alpha}{\partial x^\kappa} - \delta^\mu_\kappa\frac{1}{2}\frac{\partial\phi^\alpha}{\partial x^\nu}\frac{\partial\phi_\alpha}{\partial x_\nu}\right)$$ Bevarelseslikninger $\underline{\partial_\mu J^\mu_\kappa = 0}$ $\kappa = 0,1$

Ved tidstranslasjonssymmetri $\kappa = 0$: $\dfrac{\partial J^0_{(0)}}{\partial t} + \nabla\vec{J}_{(0)} = 0$ Energibevarelse

$$J^0_{(0)} = -\frac{1}{2}K\left(\frac{\partial\phi^\alpha}{\partial x_0}\frac{\partial\phi_\alpha}{\partial x^0} - \frac{\partial\phi^\alpha}{\partial x^1}\frac{\partial\phi_\alpha}{\partial x_1}\right) = -\frac{1}{2}K\sum_{\alpha=1}^{3}\left[\frac{1}{c^2}\left(\frac{\partial\phi_\alpha}{\partial t}\right)^2 + \left(\frac{\partial\phi_\alpha}{\partial x}\right)^2\right] = -\frac{K}{gc^2}[\text{energitetthet}]$$

$$J^1_{(0)} = -K\frac{\partial\phi^\alpha}{\partial x_1}\frac{\partial\phi_\alpha}{\partial x^0} = \frac{K}{c}\sum_{\alpha=1}^{3}\frac{\partial\phi^\alpha}{\partial x}\frac{\partial\phi_\alpha}{\partial t}$$ energitransport

Romtranslasjon (langs x-aksen) $\kappa = 1$ $\qquad \dfrac{\partial J^0_{(1)}}{\partial t} + \nabla\vec{J}_{(1)} = 0$ Impulsbevarelse.

$$J^0_{(1)} = -K\frac{\partial\phi_\alpha}{\partial x_0}\frac{\partial\phi^\alpha}{\partial x^1} = -\frac{K}{c}\sum_{\alpha=1}^{3}\frac{\partial\phi^\alpha}{\partial x}\frac{\partial\phi_\alpha}{\partial t}$$ impulstetthet

$$J^1_{(1)} = -\frac{K}{2}\left(\frac{\partial\phi^\alpha}{\partial x^1}\frac{\partial\phi_\alpha}{\partial x_1} - \frac{\partial\phi^\alpha}{\partial x^0}\frac{\partial\phi_\alpha}{\partial x_0}\right) = \frac{1}{2}K\sum_{\alpha=1}^{3}\left[\frac{1}{c^2}\left(\frac{\partial\phi_\alpha}{\partial t}\right)^2 + \left(\frac{\partial\phi_\alpha}{\partial x}\right)^2\right]$$ Spenning + impulsstrøm

Klaus. felt 24.8.90

Løsn

2a Virkningen ekstremalverdi langs banen. Variasjon om banen gir 0.

Varierer φ-koordinaten:

$$\frac{\delta S}{\delta\varphi} = -mc\,\frac{\delta}{\delta\varphi}\int ds = -mc\,\frac{\delta}{\delta\varphi}\int\sqrt{\left(\frac{ds}{d\tau}\right)^2}\,d\tau \equiv -mc\,\frac{\delta}{\delta\varphi}\int L\,d\tau = 0\,,\quad \tau = \text{egentiden.}$$

$$\Rightarrow \frac{\partial L}{\partial\varphi} - \frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{d\tau}\right)} = 0$$

$$\frac{\partial L}{\partial\varphi} = \frac{1}{2\sqrt{\ }}\frac{\partial}{\partial\varphi}\left(\frac{ds}{d\tau}\right)^2,\quad \frac{\partial L}{\partial\left(\frac{\partial\varphi}{d\tau}\right)} = \frac{1}{2\sqrt{\ }}\frac{\partial}{\partial\left(\frac{\partial\varphi}{\partial\tau}\right)}\left(\frac{ds}{d\tau}\right)^2 \Rightarrow \frac{1}{2\sqrt{\ }}\frac{\partial}{\partial\varphi}\left(\frac{ds}{d\tau}\right)^2 - \frac{d}{d\tau}\left(\frac{1}{2\sqrt{\ }}\frac{\partial}{\partial\left(\frac{\partial\varphi}{\partial\tau}\right)}\left(\frac{ds}{d\tau}\right)^2\right) = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial\varphi} - \frac{d}{d\tau}\frac{\partial}{\partial\left(\frac{\partial\varphi}{d\tau}\right)}\right)\left(\frac{ds}{d\tau}\right)^2 = 0 \qquad \text{da}\quad \sqrt{\ } = \text{konstant} = \frac{ds}{d\tau} = c \quad \text{langs banen.}$$

Her er
$$\left(\frac{ds}{d\tau}\right)^2 = \left(1-\frac{\varepsilon}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{\varepsilon}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\vartheta}{d\tau}\right)^2 - r^2\sin^2\vartheta\left(\frac{d\varphi}{d\tau}\right)^2$$

Gir:

$$\frac{\partial L}{\partial\varphi} \Rightarrow \frac{\partial\left(\frac{ds}{d\tau}\right)^2}{\partial\varphi} = 0 \qquad \frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial\tau}\right)} \Rightarrow -2r^2\sin^2\vartheta\cdot\frac{d\varphi}{d\tau}$$

$$\frac{\partial L}{\partial\varphi} - \frac{d}{d\tau}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial\tau}\right)} = 0 \Rightarrow -\frac{d}{d\tau}\left(2r^2\sin^2\vartheta\frac{d\varphi}{d\tau}\right) = 0 \Rightarrow r^2\sin^2\vartheta\,\frac{d\phi}{d\tau} = \text{konst} = \frac{L_z}{m}$$

$$\Rightarrow \boxed{\frac{d\phi}{d\tau} = \frac{L/m}{r^2\sin^2\vartheta}}$$

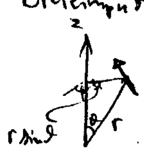
Variasjon av ϑ-koordinaten:

$$\left.\begin{aligned}\frac{\partial L}{\partial\vartheta} &\Rightarrow -2r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\tau}\right)^2\\ \frac{\partial L}{\partial\left(\frac{\partial\vartheta}{\partial\tau}\right)} &\Rightarrow -2r^2\frac{d\vartheta}{d\tau}\end{aligned}\right\} -2r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{d}{d\tau}\left(-2r^2\frac{d\vartheta}{d\tau}\right) = 0$$

Altså:

$$\boxed{\frac{d^2\vartheta}{d\tau^2} = \sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\tau}\right)^2 - \frac{2}{r}\frac{dr}{d\tau}\frac{d\vartheta}{d\tau}}$$

b) Dreieimpuls om z-aksen $L_z = m\,r\sin\vartheta\cdot r\sin\vartheta\,\frac{d\varphi}{d\tau} = \underline{m\,r^2\sin^2\vartheta\,\frac{d\varphi}{d\tau}}$



$v = r\sin\vartheta\,\frac{d\varphi}{d\tau}$

ifølge første bevegelseslikning: a) er denne konstant.

c) Annen bevegelseslikning viser at hvis vi starter med $\vartheta = \frac{\pi}{2}$ og $\frac{d\vartheta}{d\tau} = 0$ og partikkelen så vil $\frac{d^2\vartheta}{d\tau^2} = 0$ og dermed ϑ og $\frac{d\vartheta}{d\tau}$ fortsette å være 0 og bevege seg i planet $\vartheta = \frac{\pi}{2}$. Kan alltid velge koordinatsystemet slik at partikkelen starter med $\vartheta = \frac{\pi}{2}$ og at $\vartheta = \frac{\pi}{2}$ planet inneholder startretningen for partikkelen.