Klassisk felt teori Eksamen 25.01.91

1a i) $\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}$ $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = -F_{\beta\alpha}$

$$\frac{\partial \mathcal{L}}{\partial A_\kappa} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\kappa)} = -\frac{\partial}{\partial x^\mu}\left(-\frac{2}{4} F^{\alpha\beta} \frac{\partial(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}{\partial(\partial_\mu A_\kappa)}\right) = \frac{1}{2}\frac{\partial}{\partial x^\mu}\left(F^{\mu\kappa} - F^{\kappa\mu}\right) = \underline{\frac{\partial}{\partial x^\mu} F^{\mu\kappa} = 0}$$

ii) Gauge transf. $A_\mu'(x) = A_\mu(x) + \varepsilon \partial_\mu \chi(x)$ $Q_\mu = \partial_\mu \chi$

$$F_{\alpha\beta}' = \partial_\alpha A_\beta' - \partial_\beta A_\alpha' = \partial_\alpha A_\beta + \partial_\alpha \partial_\beta \chi - \partial_\beta A_\alpha - \partial_\beta \partial_\alpha \chi = \partial_\alpha A_\beta - \partial_\beta A_\alpha = F_{\alpha\beta}$$

$$\mathcal{L}' = -\frac{1}{4} F_{\alpha\beta}' F^{\alpha\beta\,\prime} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = \underline{\mathcal{L}}$$ Gauge transf er sym. transf for frie el.mag.felt.

iii) Lokal bevarelses setning:

$$\partial_\mu j^\mu = 0, \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\kappa)} Q_\kappa = \underline{-F^{\mu\kappa}\partial_\kappa \chi} \Longrightarrow \underline{\partial_\kappa(F^{\kappa\mu}\chi)} \quad \text{da } \partial_\kappa F^{\kappa\mu} = 0$$

$$\underline{\partial_\mu(F^{\mu\kappa}\partial_\kappa \chi) = 0} \quad \text{eller} \quad \underline{\partial_\mu[\partial_\kappa(F^{\kappa\mu}\chi)] = 0}$$

Global bevart størrelse:

$$\Omega = \int_V j^0 d^3x = \int_V (F^{0\kappa}\partial_\kappa \chi)\, d^3x = \int_V \partial_\kappa(F^{0\kappa}\chi)\, d^3x = \int_V \partial_k(F^{0k}\chi)\, d^3x \quad (\text{da } F^{00} = 0)$$

$$= \underline{\int_S F^{0k}\chi\, dS_k = 0} \quad \text{når } F^{0k} = 0 \text{ på overflaten } S \text{ av volumet } V.$$

Med $\chi = \text{konst} \Rightarrow \partial_\kappa \chi = 0 \Rightarrow j^\mu = 0$

Symmetrien med $\chi = \text{konst}$ ($A_\mu' = A_\mu$ ingen forandring) har som rimelig er ingen tilhörende bevarelses setning.

b i) $\mathcal{L} = \bar\psi\left(c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu - eA_\mu\right) + mc^2\right)\psi$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} = \underline{\left[c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu - eA_\mu\right) + mc^2\right]\psi = 0} \quad \text{Dirac-likn}$$

ii) $\mathcal{L}' = \bar\psi'\left(c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu - eA_\mu'\right) + mc^2\right)\psi'$ $\psi'(x) = \psi(x) e^{\frac{ie}{\hbar}\varepsilon\chi(x)} = \psi(x)\left(1 + \frac{ie}{\hbar}\varepsilon\chi + O(\varepsilon^2)\right)$

$A_\mu'(x) = A_\mu(x) + \varepsilon\partial_\mu\chi$

$$= \psi^\dagger\gamma^0 e^{-\frac{ie}{\hbar}\varepsilon\chi}\left[c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu - eA_\mu - e\varepsilon\partial_\mu\chi\right) + mc^2\right] e^{\frac{ie}{\hbar}\varepsilon\chi}\,\psi$$

$$= \psi^\dagger\gamma^0 e^{-\frac{ie}{\hbar}\varepsilon\chi}\left[e^{\frac{ie}{\hbar}\varepsilon\chi}\, c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu + \frac{\hbar}{i}\frac{i}{\hbar}e\varepsilon\,\partial_\mu\chi - eA_\mu - e\varepsilon\partial_\mu\chi\right)\psi\right]$$

$$= \underline{\psi^\dagger\gamma^0\, c\gamma^\mu\left(\frac{\hbar}{i}\partial_\mu - eA_\mu\right)\psi = \mathcal{L}}$$ Den kombinerte gaugetrans er sym. transf.

iii) $\partial_\mu j^\mu = 0$. $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} Q_{\bar\psi} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} Q_\psi = 0\cdot Q_{\bar\psi} + \bar\psi\frac{\hbar c}{i}\gamma^\mu\cdot\frac{ie}{\hbar}\chi\cdot\psi = \underline{ec\bar\psi\gamma^\mu\psi\chi}$

$\chi = \text{konst}$. $\partial_\mu j^\mu = \chi\,\partial_\mu(ec\bar\psi\gamma^\mu\psi) = 0$ Bevart strøm: $j^{\mu\prime} = \underline{ec\bar\psi\gamma^\mu\psi}$

Total ladning: $\frac{d}{dt}\int_V \frac{j^0}{c} d^3x = -\int_V \partial_k j^k d^3x = -\int_S j^k dS^k = 0$

$$Q = \int_V \rho\, d^3x = \chi c\int_V e\bar\psi\gamma^0\psi\, d^3x = \text{konst} \Rightarrow \underline{\Omega' = \int \bar\psi e\gamma^0\psi\, d^3x = \text{konst}}$$

Klass. Felt 23.01.91

2a) Koordinater $dx^0 = c\,dt$, $dx^1 = dr$ $dx^2 = d\vartheta$ $dx^3 = d\varphi$

Intervall likningen i $ds^2 = c^2dt^2 - \frac{(R(t))^2}{1-kr^2}dr^2 - (R(t))^2r^2d\vartheta^2 - (R(t))^2r^2\sin^2\vartheta\,d\varphi^2$

Direkte fra intervall likningen

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{R(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -R(t)^2r^2 & 0 \\ 0 & 0 & 0 & -R(t)^2r^2\sin^2\vartheta \end{pmatrix}$$

Sammenheng mellom $g_{\mu\nu}$ og $g^{\mu\nu}$:

$dx_\mu = g_{\mu\nu}dx^\nu = g_{\mu\nu}g^{\nu\lambda}dx_\lambda \Rightarrow g_{\mu\nu}g^{\nu\lambda} = \delta_\mu{}^\lambda \Rightarrow \underline{g_{\nu\mu}g^{\nu\lambda} = \delta_\mu{}^\lambda}$ da $g_{\mu\nu} = g_{\nu\mu}$

Da $g_{\mu\nu}$ bare har diagonale element får vi $g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{R(t)^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{R(t)^2r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{R(t)^2r^2\sin^2\vartheta} \end{pmatrix}$$

Ser at hvis $dr = 0$ $d\vartheta = 0$ $d\varphi = 0$ (dvs **fast** i koordinatsystemet, følger med koord.syst.)

$ds^2 = c^2d\tau^2 = c^2dt^2 \Rightarrow dt = d\tau$

Tiden t her er altså <u>egentiden</u> for en klokke som ligger fast i koordinatsystem på det aktuelle stedet.

b) $\Gamma^1_{01} = \frac{1}{2}g^{11}\left(\frac{\partial g_{10}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^1}\right) = \frac{1}{2}g^{11}\frac{\partial g_{11}}{\partial x^0}$ Da bare $g^{\mu\mu}$ og $g_{\mu\mu} \neq 0$

$= \frac{1}{2}\left(-\frac{1-kr^2}{R^2}\right)\left(-\frac{2R\frac{dR}{dx^0}}{1-kr^2}\right) = \frac{1}{2}g^{11}\,2g_{11}\frac{\dot R}{R} = \underline{\underline{\frac{\dot R}{R}}}$, $\dot R = \frac{dR(t)}{dx^0} = \frac{1}{c}\frac{dR(t)}{dt}$

$R_{00} = \underbrace{\frac{\partial\Gamma^k_{0k}}{\partial x^k}}_{0} - \frac{\partial\Gamma^k_{0k}}{\partial x^0} + \underbrace{\Gamma^k_{gk}\Gamma^g_{00}}_{0} - \Gamma^k_{g0}\Gamma^g_{0k} = -\frac{\partial\Gamma^k_{0k}}{\partial x^0} - \Gamma^k_{j0}\Gamma^j_{0k}$ da $\Gamma^k_{00} = 0$

$= -\frac{\partial}{\partial x^0}\left(\frac{\dot R}{R}\right)3 - \left(\frac{\dot R}{R}\right)^2 3 = -3\frac{\ddot R}{R} + 3\frac{(\dot R)^2}{R^2} - 3\left(\frac{\dot R}{R}\right)^2 = \underline{\underline{-3\frac{\ddot R}{R}}}$

$\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij} = 1\cdot\left(-3\frac{\ddot R}{R}\right) + \underbrace{g^{ij}(-g_{ij})}_{-\delta^i{}_j = -3}\left(\frac{\ddot R}{R} + 2\frac{\dot R^2}{R^2} + 2\frac{k}{R^2}\right)$

$= \underline{-6\left(\frac{\ddot R}{R} + \frac{\dot R^2}{R^2} + \frac{k}{R^2}\right)}$

c) $R_{00} - \frac{1}{2}g_{00}\mathcal{R} = -3\frac{\ddot R}{R} + 3\left(\frac{\ddot R}{R} + \frac{\dot R^2}{R^2} + \frac{k}{R^2}\right) = 3\left(\frac{\dot R}{R}\right)^2 + 3\frac{k}{R^2} = \frac{8\pi G}{c^4}T_{00} = \frac{8\pi G}{c^4}\rho c^2$

$\underline{\dot R^2 = \frac{8\pi G}{3c^2}\rho R^2 - k}$

Med totale masse innenfor $R(t)$ konstant: Setter $\frac{4\pi}{3}R(t)^3\rho(t)^3 = M$ = konst.

$\underline{R\dot R^2 = 2MG - kR}$

$k=0$ $\sqrt{R}\,dR = \sqrt{2MG}\,d(ct) \Rightarrow \frac{2}{3}R^{3/2} = \sqrt{2MG}\,ct$ når velger $R=0$ for $t=0$

$\underline{R(t) = \left(\frac{3}{2}c\right)^{2/3}(2MG)^{1/3}t^{2/3}}$ Vokser hele tiden som $t^{2/3}$

$k>0$ $\dot R(t)^2 = \frac{2MG}{R} - k$ For $R(t)$ liten vil $\dot R(t)^2 > 0$ og med $\dot R(t) > 0$ vil $R(t)$ vokse til $R(t_m) = \frac{2MG}{k} = R_m$ Da $\dot R(t) = 0$

og må enten $R(t) = R_m$ for alle $t > t_m$ eller minke $\dot R(t) < 0$.

For små $R(t)$ vil ikke approksimasjonen $p=0$ og ideell gass gjelde.