Klassisk feltteori. Eks 14.1.92

Løsninger

1 a i) $\mathcal{L} = -\frac{1}{2} G^*_{\mu\nu} G^{\mu\nu} + k^2 W^*_\mu W^\mu \qquad G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$

$$\frac{\partial \mathcal{L}}{\partial W^*_\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial(\partial_\nu W^*_\mu)} = k^2 W^\mu - \partial_\nu\left(-\tfrac{1}{2}(G^{\nu\mu} - G^{\mu\nu})\right) = \underline{\partial_\nu G^{\nu\mu} + k^2 W^\mu = 0}$$

$$\frac{\partial \mathcal{L}}{\partial W_\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial(\partial_\nu W_\mu)} = \underline{k^2 W^{\mu *} + \partial_\nu G^{\nu\mu *} = 0}$$

ii) $$T_K^{\mu\nu} = \partial^\mu W_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\nu W_\lambda)} + \partial^\mu W^*_\lambda \frac{\partial \mathcal{L}}{\partial(\partial_\nu W^*_\lambda)} - g^{\mu\nu}\mathcal{L}$$

$$= \underline{\partial^\mu W_\lambda G^{\lambda\nu *} + \partial^\mu W^*_\lambda G^{\lambda\nu} - g^{\mu\nu}\mathcal{L}}$$ ikke sym. i $\mu\nu$

iii) Symmetriseres med divergens av antisym. tensor: $\partial_\lambda(W^\mu G^{\nu\lambda *} + W^{\mu *} G^{\nu\lambda}) =$

$$= (\partial_\lambda W^\mu) G^{\nu\lambda *} + (\partial_\lambda W^{\mu *}) G^{\nu\lambda} + W^\mu \partial_\lambda G^{\nu\lambda *} + W^{\mu *}\partial_\lambda G^{\nu\lambda}$$

$$= (\partial_\lambda W^\mu) G^{\nu\lambda *} + (\partial_\lambda W^{\mu *}) G^{\nu\lambda} + W^\mu k^2 W^{\nu *} + W^{\mu *} k^2 W^\nu$$

$$T_B^{\mu\nu} = T_K^{\mu\nu} + \partial_\lambda(W^\mu G^{\nu\lambda *} + W^{\mu *} G^{\nu\lambda})$$

$$= \partial^\mu W_\lambda G^{\lambda\nu *} + \partial_\lambda W^\mu G^{\nu\lambda *} + \partial^\mu W^*_\lambda G^{\lambda\nu} + \partial_\lambda W^{\mu *} G^{\nu\lambda} + k^2(W^\mu W^{\nu *} + W^{\mu *} W^\nu) - g^{\mu\nu}\mathcal{L}$$

$$= \underline{G^\mu{}_\lambda G^{\lambda\nu *} + G^{\mu *}{}_\lambda G^{\lambda\nu} + k^2(W^\mu W^{\nu *} + W^{\mu *} W^\nu) - g^{\mu\nu}\mathcal{L}}$$

iv) Dreieimpulsbevarelse

$$\partial_\lambda \mathcal{M}^{\lambda\mu\nu} = \frac{1}{c}\partial_\lambda(x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda})$$

$$= \frac{1}{c}(\delta^\mu_\lambda T^{\nu\lambda} - \delta^\nu_\lambda T^{\mu\lambda} - x^\mu \underbrace{\partial_\lambda T^{\lambda\nu}}_{0} + x^\nu \underbrace{\partial_\lambda T^{\lambda\mu}}_{0})$$

$$= \frac{1}{c}(T^{\nu\mu} - T^{\mu\nu}) = 0 \quad \text{når } T^{\mu\nu} \text{ er symmetrisk.}$$

Belinfante tensoren er sym, Kanoniske ikke.

$$\partial_\lambda\left(\mathcal{M}^{\lambda\mu\nu} + \tfrac{1}{c}\Psi^{\nu\mu\lambda} - \tfrac{1}{c}\Psi^{\mu\nu\lambda}\right) = \frac{1}{c}\left(T_K^{\nu\mu} + \partial_\lambda \Psi^{\nu\mu\lambda} - (T_K^{\mu\nu} + \partial_\lambda \Psi^{\mu\nu\lambda})\right) = \frac{1}{c}(T_B^{\mu\nu} - T_B^{\nu\mu}) = 0$$

Tillegg: Indre spinn:

$$\underline{\mathcal{S}^{\lambda\mu\nu} = \frac{1}{c}(\Psi^{\nu\mu\lambda} - \Psi^{\mu\nu\lambda}) = \frac{1}{c}[W^\mu G^{\nu\lambda *} + W^{\mu *} G^{\nu\lambda} - W^\nu G^{\mu\lambda *} - W^{\nu *} G^{\mu\lambda}]}$$

$$\underline{\mathcal{S}^{0k\ell} = \frac{1}{c}[W^k G^{\ell 0 *} + W^{k *} G^{\ell 0} - W^\ell G^{k 0 *} - W^{\ell *} G^{k0}]}$$

b) Global transformasjon:
$W'_\mu = e^{i\alpha} W_\mu$ (α = konstant) $\Rightarrow$ $G^{*\prime}_{\mu\nu} G^{\mu\nu\prime} = e^{-i\alpha} e^{i\alpha} G^*_{\mu\nu} G^{\mu\nu} = G^*_{\mu\nu} G^{\mu\nu}$

$k^2 W^{*\prime}_\mu W^{\mu\prime} = k^2 e^{-i\alpha} e^{i\alpha} W^*_\mu W^\mu = k^2 W^*_\mu W^\mu$

$$\Rightarrow \underline{\mathcal{L}' = \mathcal{L}}$$

Lorentzbetingelsen må være oppfylt:
Fra feltlikningene:

$$\partial_\mu(\partial_\nu \partial^\nu W^\mu - \partial_\nu \partial^\mu W^\nu + k^2 W^\mu) = \partial_\nu \partial^\nu \partial_\mu W^\mu - \partial_\mu \partial^\mu \partial_\nu W^\nu + k^2 \partial_\mu W^\mu =$$

$$= \underline{k^2 \partial_\mu W^\mu = 0}$$

c) Lokal gauge transf.
$W'_\mu = e^{i\alpha(x)} W_\mu$ med $\alpha(x)$. $\Rightarrow$ $W^{*\prime}_\mu W^{\mu\prime} = W^*_\mu e^{-i\alpha} e^{i\alpha} W^\mu = W^*_\mu W^\mu$ Invariant

$$G^{\mu\nu\prime} = \partial^\mu W^{\nu\prime} - \partial^\nu W^{\mu\prime} = e^{i\alpha}(\partial^\mu W^\nu - \partial^\nu W^\mu) + i\, e^{i\alpha}((\partial^\mu\alpha) W^\nu - (\partial^\nu\alpha) W^\mu) \neq G^{\mu\nu}$$

i) Kobler til felt $\partial^\mu W^\nu \to \underline{(\partial^\mu - iB^\mu) W^\nu}$ (Minimal kobling)

$$G^{\mu\nu\prime} = (\partial^\mu - iB^{\mu\prime}) W^{\nu\prime} - (\partial^\nu - iB^{\nu\prime}) W^{\mu\prime} = e^{i\alpha}[(\partial^\mu - iB^{\mu\prime}) W^\nu - (\partial^\nu - iB^{\nu\prime}) W^\mu + i(\partial^\mu\alpha W^\nu - \partial^\nu\alpha W^\mu)]$$

ii) Tilleggsleddene forsvinner hvis $\underline{B^{\mu\prime} = B^\mu + \partial^\mu\alpha}$

$$G^{\mu\nu\prime} = e^{i\alpha}[(\partial^\mu - iB^\mu) W^\nu - (\partial^\nu - iB^\nu) W^\mu] = e^{i\alpha} G^{\mu\nu} \quad \text{og} \quad G^{*\mu\nu\prime} G'_{\mu\nu} = G^{*\mu\nu} G_{\mu\nu}$$

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2a) Egentid for observatör på fast plass i koordinatsystemet : $dx^k = 0$

$$c^2 d\tilde{t}^2 = g_{00}\, dx^0 dx^0 = g_{00} c^2 dt^2 \qquad d\tilde{t} = \sqrt{g_{00}}\, dt$$

Endelig tidsrom $\quad \Delta\tilde{t} = \tilde{t}_2 - \tilde{t}_1 = \int_{\tilde{t}_1}^{\tilde{t}_2} d\tilde{t} = \int_{t_1}^{t_2} \sqrt{g_{00}}\, dt = \sqrt{g_{00}}\,(t_2 - t_1)$ hvis $\frac{\partial g_{00}}{\partial t} = 0$

b) I gravitasjonsfeltet rundt en kulesymmetrisk masse i ro er intervallet gitt ved

$$ds^2 = \left(1 - \frac{\varepsilon}{r}\right) c^2 dt^2 - \left(1 - \frac{\varepsilon}{r}\right)^{-1} dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \qquad \varepsilon = \frac{2GM}{r}$$

En partikkel beveger seg i dette feltet slik at intervallet blir en ekstremal verdi :

$$\int ds = \int \frac{ds}{d\sigma} d\sigma = \min \;\hat{=}\; \int \mathcal{L}\, d\sigma \;\Rightarrow\; \frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\sigma} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial x^\mu}{\partial \sigma}\right)} = 0$$

$$\mathcal{L} = \left(1 - \frac{\varepsilon}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{1 - \frac{\varepsilon}{r}} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left[\left(\frac{d\vartheta}{d\tau}\right)^2 + \sin^2\vartheta \left(\frac{\partial\varphi}{\partial\tau}\right)^2\right] = c^2$$

$$\frac{\partial \mathcal{L}}{\partial t} - \frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial t}{\partial \tau}\right)} = -\frac{d}{d\tau}\left(2c^2 \left(1 - \frac{\varepsilon}{r}\right) \frac{dt}{d\tau}\right) = 0 \Rightarrow \frac{dt}{d\tau} = \frac{A}{1 - \frac{\varepsilon}{r}}$$

$$\underline{u^0 = \frac{d(ct)}{d\tau} = \frac{Ac}{1 - \frac{\varepsilon}{r}}}$$

Fra intervallformelen $\quad \vartheta = \vartheta_0,\ \varphi = \varphi_0$

$$\left(1 - \frac{\varepsilon}{r}\right) c^2 \left(\frac{A}{1 - \frac{\varepsilon}{r}}\right)^2 - \frac{1}{1 - \frac{\varepsilon}{r}} \left(\frac{dr}{d\tau}\right)^2 = c^2$$

$$u^r = \frac{dr}{d\tau} = -c \left[\left(\frac{A^2}{1 - \frac{\varepsilon}{r}} - 1\right)\left(1 - \frac{\varepsilon}{r}\right)\right]^{1/2} = \underline{-c\left(A^2 - 1 + \frac{\varepsilon}{r}\right)^{1/2}}$$

Fortegn - ved fall innover

Begynnelsesbetingelse $u^r = 0$ ved $r = R$: $\quad A^2 - 1 + \frac{\varepsilon}{R} = 0$

$$\underline{u^r = -c\left(\frac{\varepsilon}{r} - \frac{\varepsilon}{R}\right)^{1/2}} \qquad \underline{u^0 = c\, \frac{\left(1 - \frac{\varepsilon}{R}\right)^{1/2}}{1 - \frac{\varepsilon}{r}}}$$

c) $R = \infty$

$$\frac{dr}{d\tau} = -c\sqrt{\frac{\varepsilon}{r}} \Rightarrow \sqrt{r}\, dr = -c\sqrt{\varepsilon}\, d\tau \Rightarrow \frac{2}{3}\left(r^{3/2} - r_0^{3/2}\right) = -c\sqrt{\varepsilon}\,(\tau - \tau_0)$$

Velger $\tau = 0$ når $r = r_0 \Rightarrow \tau_0 = 0 \Rightarrow$ $\quad \underline{r = \left(r_0^{3/2} - \frac{3}{2} c\sqrt{\varepsilon}\, \tau\right)^{2/3}}$

d) I observatørens egensystem :

$$\frac{d\tilde{r}}{d\tilde{t}} = \frac{\sqrt{-g_{rr}}\, dr}{\sqrt{g_{00}}\, dt} = \sqrt{\frac{\left(1 - \frac{\varepsilon}{r}\right)^{-1}}{1 - \frac{\varepsilon}{r}}}\, \frac{dr}{dt} = \frac{1}{1 - \frac{\varepsilon}{r}} \frac{dr}{dt} \qquad (g_{0r} = 0)$$

$$\frac{dr}{dt} = \frac{dr}{d\tau}\frac{d\tau}{dt} = u^r \left(\frac{u^0}{c}\right)^{-1} = -c\sqrt{\frac{\varepsilon}{r}}\left(1 - \frac{\varepsilon}{r}\right)$$

Innsatt:

$$\frac{d\tilde{r}}{d\tilde{t}} = \frac{1}{1 - \frac{\varepsilon}{r}}\left(-c\sqrt{\frac{\varepsilon}{r}}\right)\left(1 - \frac{\varepsilon}{r}\right) = \underline{\underline{-c\sqrt{\frac{\varepsilon}{r}}}} \to -c \text{ når } r \to \varepsilon$$

med $r = r_0$ $\quad \underline{\underline{\frac{d\tilde{r}}{d\tilde{t}} = -c\sqrt{\frac{\varepsilon}{r_0}}}}$