Klassisk feltteori Eksamen 20.1.93

Løsninger

1a) Hamiltons variasjonsprinsipp

$$\delta S = \delta \int \mathcal{L} d^4x = \int \left[\frac{\partial \mathcal{L}}{\partial \varphi_n} \delta\varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \delta(\partial_\mu \varphi_n) \right] d^4x$$

Partiell integrasjon av siste leddet gir:

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi_n} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \right] \delta\varphi_n \, d^4x + \int_{\text{overflate}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \underbrace{\delta\varphi_n}_{0} dS^\mu = 0$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi_n} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \varphi_n}{\partial x^\mu}\right)} = 0$$

når variasjonen $\delta\varphi_n$ er fri bare med $\delta\varphi_n = 0$ på overflaten.

1b)

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \partial^\nu \varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \partial_\mu \partial^\nu \varphi_n - g^{\mu\nu} \frac{d\mathcal{L}}{dx^\mu}$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi_n} \partial^\nu \varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} \partial_\mu \partial^\nu \varphi_n - g^{\mu\nu} \left(\frac{\partial \mathcal{L}}{\partial \varphi_n} \partial_\mu \varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\rho \varphi_n)} \partial_\rho \partial_\mu \varphi_n \right) = \underline{0}$$

når $\mathcal{L}$ ikke avhenger eksplitt av x^μ.

For $\nu = 0$ energibevarelse

$$\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{k0}}{\partial x^k} = \frac{1}{c} \underline{\left(\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot \vec{S} \right)} = 0$$

med $T^{00} = \mathcal{H}$ energitetthet, $T^{k0} = \frac{1}{c} S^k$ Poynting vektor

For $\nu = l$ impulsbevarelse

$$\frac{\partial T^{0l}}{\partial x^0} + \frac{\partial T^{kl}}{\partial x^k} = \underline{\frac{\partial g^l}{\partial t} - \frac{\partial}{\partial x^k} \sigma^{kl}} = 0$$

med $T^{0l} = c g^l$ impulstetthet, $T^{kl} = -\sigma^{kl}$ {spenning og impulsstrøm

1c) Dreieimpulstetthet $\mathcal{M}^0_i$ er en av komponentene

$$\mathcal{M}^\mu_i = \frac{1}{c} \varepsilon_{ijk} x^j T^{\mu k} \qquad \mu = 0,1,2,3$$

Lokal bevarelsessetning

$$\frac{\partial \mathcal{M}^\mu_i}{\partial x^\mu} = \frac{1}{c} \varepsilon_{ijk} \delta^j_\mu T^{\mu k} + \frac{1}{c} \varepsilon_{ijk} x^j \underbrace{\frac{\partial T^{\mu k}}{\partial x^\mu}}_{0} \quad \text{når impulsbevarelse}$$

$$= \frac{1}{c} \frac{1}{2} \left(\varepsilon_{ijk} T^{jk} + \varepsilon_{ikj} T^{kj} \right)$$

$$= \frac{1}{2c} \varepsilon_{ijk} \left(T^{jk} - T^{kj} \right) = \underline{0} \quad \text{når } T^{jk} = T^{kj}$$

Utskrevet

$$\frac{\partial \mathcal{M}^\mu_i}{\partial x^\mu} = \frac{1}{c} \frac{\partial}{\partial t} \left(x^j \frac{1}{c} T^{0k} - x^k \frac{1}{c} T^{0j} \right) + \partial_l \left(x^j \frac{1}{c} T^{lk} - x^k \frac{1}{c} T^{lj} \right) = 0$$

$$\frac{1}{c} \left[\frac{\partial}{\partial t} (x^j g^k - x^k g^j) - \frac{\partial}{\partial x^l} (x^j \sigma^{lk} - x^k \sigma^{lj}) \right] = 0$$

$$\underline{\frac{\partial}{\partial t} M^0_i + \frac{\partial}{\partial x^l} D^l_i = 0} \qquad i,j,k = 1,2,3 \text{ syklisk.}$$

1d) For total energi og impuls i hele rommet

$$0 = \int_{\text{hele rommet}} \left(\frac{\partial T^{0\nu}}{\partial x^0} + \frac{\partial T^{k\nu}}{\partial x^k} \right) d^3x = \frac{\partial}{\partial t} \left(\frac{1}{c} \int_{\text{volum}} T^{0\nu} d^3x \right) + \int_{\text{overflaten}} \underbrace{T^{k\nu}}_{0} dS_k = \frac{1}{c} \frac{d}{dt} \int T^{0\nu} d^3x = 0$$

0 da $\varphi = 0$ [illegible] dermed $T^{k\nu} = 0$ på overflaten

Konstant total energi $\int_{\text{volum}} T^{00} d^3x = \underline{E = \text{konst.}}$

og impulskomponenter $\int_{\text{volum}} T^{0k} d^3x = \underline{G^k = \text{konstant}}$

Likedan: Total dreieimpuls komponent om i-aksen er bevart.

$$\underline{\int_{\text{hele rommet}} M^0_i d^3x = \text{konst}}$$

Kvante 2 20.1.93

1e i) Klassisk: Invariant ved rotasjon, inversjon og forskyvning i 3-dim rom

$$x^{k'} = \alpha^{k'}{}_{\ell}\, x^{\ell} + d^{k'} \qquad \text{med} \qquad \alpha^{k'}{}_{\ell}\, \alpha_{k'}{}^{m} = \delta_{\ell}{}^{m} \qquad k = 1,2,3$$

$$t' = t + t_0$$

1e ii) Relativistisk. Invariant ved transformasjonene i e i) og i tillegg ved Lorentz-transformasjon og tidsinversjon.

$$x^{\mu'} = \alpha^{\mu'}{}_{\nu}\, x^{\nu} + d^{\mu'} \qquad \alpha^{\mu'}{}_{\nu}\, \alpha_{\mu'}{}^{\lambda} = \delta_{\nu}{}^{\lambda} \qquad \mu = 0,1,2,3$$

1f) u^k og $\frac{\partial}{\partial x^{\ell}}$ transformerer som vektorer. t er skalar

$u^k u^k$, $\frac{\partial u^k}{\partial x^k}$, $\frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^i}$, $\frac{\partial u^k}{\partial t}\frac{\partial u^k}{\partial t}$ er derfor alle skalarer og dermed er $\mathcal{L}$ en skalar (i 3-dim rom) d.v.s invariant.

Andre krav:

Superposisjonsprinsippet skal gjelde for små deformasjoner d.v.s feltlikningene må være lineære i u^k og $\mathcal{L}$ høyst kvadratisk i u^k

Begynnelsesbetingelsene: Gitt feltet og dets tidsderiverte ved $t = t_0$ skal bestemme feltets videre utvikling. d.v.s feltlikningene må inneholde $\frac{\partial^2 u^k}{\partial t^2}$ og $\mathcal{L}$ høyst $\frac{\partial u^k}{\partial t}$

1g)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u^k)}\frac{\partial u^k}{\partial x^\nu} - \delta^{\mu\nu}\mathcal{L} \qquad \text{Bruker } x^0 = t$$

$$T^{00} = -\rho \frac{\partial u^k}{\partial t}\frac{\partial u^k}{\partial t} - \mathcal{L} = \underline{-\left(\tfrac{1}{2}\rho \frac{\partial u^k}{\partial t}\frac{\partial u^k}{\partial t} + \tfrac{1}{2}\mu \frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^i} + \tfrac{1}{2}(\mu+\lambda)\frac{\partial u^i}{\partial x^i}\frac{\partial u^k}{\partial x^k}\right)}$$

$$= -(\text{energitetthet}) = -\mathcal{H}$$

$$T^{i0} = \underline{\tfrac{1}{c}\left(\mu \frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial t} + (\mu+\lambda)\frac{\partial u^k}{\partial x^k}\frac{\partial u^i}{\partial t}\right) = -\tfrac{1}{c}S^i}$$

$$T^{0j} = \underline{-\rho c \frac{\partial u^k}{\partial t}\frac{\partial u^k}{\partial x^j} = -c\, g^j}$$

$$T^{ij} = \underline{\mu \frac{\partial u^k}{\partial x^i}\frac{\partial u^k}{\partial x^j} + (\mu+\lambda)\frac{\partial u^k}{\partial x^k}\frac{\partial u^i}{\partial x^j} - \delta^{ij}\mathcal{L}}$$

Den kanoniske energi-impulstensor her er ikke symmetrisk

$$T^{ij} \neq T^{ji} \qquad T^{0i} \neq T^{i0}$$

d.v.s. systemets dreieimpuls er ikke bevart alene men trenger et indre spinn i tillegg. For å få bevarelse.

Klassisk feltteori 20.1.93

2 i) Lyset går korteste (ekstremal) vei i tidrommet

$$\delta \int ds = \delta \int \sqrt{\left(\frac{ds}{d\sigma}\right)^2}\, d\sigma = 0 \qquad \left(\text{eller også } \delta \int \left(\frac{ds}{d\sigma}\right)^2 d\sigma = 0\right)$$

med $\left(\frac{ds}{d\sigma}\right)^2 = \left(1 - \frac{\varepsilon}{r}\right) c^2 \left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{1-\frac{\varepsilon}{r}}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left[\left(\frac{d\vartheta}{d\sigma}\right)^2 + \sin^2\vartheta \left(\frac{d\varphi}{d\sigma}\right)^2\right]$

Gir bevegelses likningene

$$\frac{\partial \sqrt{\left(\frac{ds}{dt}\right)^2}}{\partial t} - \frac{d}{d\sigma}\frac{\partial \sqrt{\ }}{\partial\left(\frac{\partial t}{\partial \sigma}\right)} = 0 \quad \Rightarrow \quad -\frac{d}{d\sigma}\left(\frac{1}{\sqrt{\ }}\left(1-\frac{\varepsilon}{r}\right)c^2 \frac{dt}{d\sigma}\right) = 0$$

$$\frac{\partial \sqrt{\ }}{\partial \varphi} - \frac{d}{d\sigma}\frac{\partial \sqrt{\ }}{\partial\left(\frac{\partial \varphi}{\partial \sigma}\right)} = 0 \quad \Rightarrow \quad -\frac{d}{d\sigma}\left(\frac{1}{\sqrt{\ }}\, r^2 \sin^2\vartheta \frac{d\varphi}{d\sigma}\right) = 0$$

Gir $c\left(1 - \frac{\varepsilon}{r}\right)\frac{dt}{d\sigma} = \text{konst} = A \quad \Rightarrow \quad \frac{dt}{d\sigma} = \frac{A/c}{1-\frac{\varepsilon}{r}}$

$r^2 \sin^2\vartheta \frac{d\varphi}{d\sigma} = \text{konst} = B \quad \Rightarrow \quad \frac{d\varphi}{d\sigma} = \frac{B}{r^2}$ når $\vartheta = \frac{\pi}{2}$

ii) Fra intervall likningen som for en lysstråle $ds = 0$ er

$$\left(1 - \frac{\varepsilon}{r}\right)c^2\left(\frac{dt}{d\sigma}\right)^2 - \frac{1}{1-\frac{\varepsilon}{r}}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\varphi}{d\sigma}\right)^2 = 0$$

fåer ved innsetting

$$\left(1-\frac{\varepsilon}{r}\right)c^2 \frac{\frac{1}{c^2}A^2}{\left(1-\frac{\varepsilon}{r}\right)^2} - \frac{1}{1-\frac{\varepsilon}{r}}\left(\frac{dr}{d\sigma}\right)^2 - r^2\frac{B^2}{r^4} = 0$$

$$\underline{\left(\frac{dr}{d\sigma}\right)^2 = A^2 - \frac{B^2}{r^2}\left(1-\frac{\varepsilon}{r}\right)}$$