Eksamen 15.1.96 Klassisk felt teori . Løsninger

I a) $\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} = -\kappa^2\varphi - \frac{\partial}{\partial x^\mu}\frac{\partial \varphi}{\partial x_\mu} = 0$ $\quad \underline{\underline{\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + \kappa^2\right)\varphi = 0}}$

b) $\mathcal{L}$ inneholder ingen koordinat eksplisitt $\frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \Rightarrow$ Bevarelsessetning for hver x^μ.

Utleder:

$$\frac{d\mathcal{L}}{dx^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial \varphi}\frac{\partial \varphi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\frac{\partial(\partial_\nu\varphi)}{\partial x^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\frac{\partial \varphi}{\partial x^\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\frac{\partial(\partial_\mu\varphi)}{\partial x^\nu}$$ (Fra E-L-likn.)

$$= \underline{\frac{\partial \mathcal{L}}{\partial x^\mu} + \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\frac{\partial \varphi}{\partial x^\mu}\right)}$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\frac{\partial \varphi}{\partial x^\mu} - \delta_\mu^\nu \mathcal{L}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

$$\underline{\frac{\partial}{\partial x^\nu} T^\nu{}_\mu = 0} \quad \text{med} \quad T^\nu{}_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi)}\partial_\mu\varphi - \delta_\mu^\nu \mathcal{L}$$

eller $\underline{\underline{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - g^{\mu\nu}\mathcal{L} \quad \text{med} \quad \frac{\partial T^{\mu\nu}}{\partial x^\mu} = 0}}$ Med minkowski-metrikken $g^{\mu\nu} = \begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$

c) Med det gitte (Klein-Gordon) feltet:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial \varphi}{\partial x_\mu}, \qquad T^{\mu\nu} = \frac{\partial \varphi}{\partial x_\mu}\frac{\partial \varphi}{\partial x_\nu} - g^{\mu\nu}\frac{1}{2}\left(\frac{\partial \varphi}{\partial x^\mu}\frac{\partial \varphi}{\partial x_\mu} - \kappa^2\varphi^2\right)$$

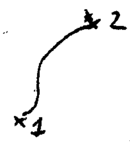
$$T^{00} = \frac{\partial \varphi}{\partial x_0}\frac{\partial \varphi}{\partial x_0} - g^{00}\frac{1}{2}\left(\frac{\partial \varphi}{\partial x^0}\frac{\partial \varphi}{\partial x^0} - \frac{\partial \varphi}{\partial x^k}\frac{\partial \varphi}{\partial x^k} - \kappa^2\varphi^2\right)$$

$$= \frac{1}{2}\left(\frac{1}{c^2}\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x^k}\frac{\partial \varphi}{\partial x^k} + \kappa^2\varphi^2\right) = \frac{1}{2}\left(\frac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + (\nabla\varphi)^2 + \kappa^2\varphi^2\right) = \underline{\mathcal{H}}$$ energitetthet

$$T^{k0} = \frac{\partial \mathcal{L}}{\partial(\partial_k\varphi)}\frac{\partial \varphi}{\partial x_0} = \frac{\partial \varphi}{\partial x_k}\frac{1}{c}\frac{\partial \varphi}{\partial t} = -\frac{1}{c}\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial x^k} = \underline{\frac{1}{c}S^k}$$ energistrømtetthet

$$\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{k0}}{\partial x^k} = \frac{1}{c}\frac{\partial \mathcal{H}}{\partial t} + \frac{1}{c}\frac{\partial S^k}{\partial x^k} = \underline{\underline{\frac{1}{c}\left(\frac{\partial \mathcal{H}}{\partial t} + \nabla\cdot\vec{S}\right) = 0}}$$ $\underline{\text{lokal energibevarelse}}$

II a) Virkningsintegralet langs en banen $x^\mu(\sigma)$ fra 1 til 2



$$S[x(\sigma)] = \int_1^2 ds = \int_1^2 \frac{ds}{d\sigma}d\sigma = \int_1^2 \sqrt{\left(\frac{ds}{d\sigma}\right)^2}d\sigma \equiv \int_1^2 L\, d\sigma$$

med $L = \sqrt{\left(\frac{ds}{d\sigma}\right)^2} = \sqrt{g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}$ (med baneparameter σ)

har ekstremalverdi for baner som oppfyller Euler-Lagrange ligningen

$$\frac{\partial L}{\partial x^\nu} - \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{dx^\nu}{d\sigma}\right)} = 0$$

$$\frac{\partial L}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(\sqrt{\left(\frac{ds}{d\sigma}\right)^2}\right) = \frac{1}{\sqrt{\ }}\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}$$

$$\frac{\partial L}{\partial\left(\frac{dx^\nu}{d\sigma}\right)} = \frac{1}{\sqrt{\ }}\left(g_{\nu\beta}\frac{dx^\beta}{du} + g_{\alpha\nu}\frac{dx^\alpha}{d\sigma}\right)$$

Løsn. Klass Felt 15.1.96

Gir $\frac{1}{\sqrt{\ }} \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} - \frac{d}{d\sigma}\left(\frac{1}{\sqrt{\ }}\left(g_{\nu\beta}\frac{dx^\beta}{d\sigma} + g_{\alpha\nu}\frac{dx^\alpha}{d\sigma}\right)\right) = 0$

Med $\sigma = s = c\tau$ blir $\sqrt{\ } = 1$ og $\frac{d}{d\sigma} = \frac{\partial}{\partial x^\alpha}\frac{dx^\alpha}{d\sigma}$:

$$\frac{\partial g_{\alpha\beta}}{\partial x^\nu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{\partial g_{\nu\beta}}{\partial x^\alpha}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{\partial g_{\alpha\nu}}{\partial x^\beta}\frac{dx^\beta}{ds}\frac{dx^\alpha}{ds} - g_{\nu\beta}\frac{d^2x^\beta}{ds^2} - g_{\alpha\nu}\frac{d^2x^\alpha}{ds^2} = 0$$

$$2\, g_{\nu\beta}\frac{d^2x^\beta}{ds^2} + \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

Benytter $g^{\nu\mu} g_{\nu\beta} = \delta^\mu_\beta$

$$\frac{d^2x^\mu}{ds^2} + g^{\nu\mu}\,\Gamma_{\nu\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \quad \text{med} \quad \Gamma_{\nu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right)$$

$$\underline{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0} \qquad \tau = \frac{s}{c} \text{ egen-tiden}$$

b) Parallellforskyver hastighetsvektoren $V^\mu = \frac{dx^\mu}{d\tau}$ ett stykke dx^β

Forandring: $dV^\mu = -\,K^\mu_{\alpha\beta}\, V^\alpha dx^\beta$

eller forandring pr tidsenhet når $dx^\beta = V^\beta d\tau$

$$\underline{\frac{dV^\mu}{d\tau} + K^\mu_{\alpha\beta}\, V^\alpha V^\beta = 0}$$

Sammenlikning med a) (Samme bevegelseslikning) gir $\underline{\underline{K^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta}}}$

c) I nye koordinatsystem vil bevegelseslikningen se ut som i det gamle

Trenger $\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \frac{1}{2}\Gamma^\mu_{\nu\beta}(P)\,[x^\beta - x^\beta(P)] + \frac{1}{2}\Gamma^\mu_{\alpha\nu}(P)\,[x^\alpha - x^\alpha(P)] =$

$= \underline{\delta^\mu_\nu + \Gamma^\mu_{\alpha\nu}(P)\,[x^\alpha - x^\alpha(P)]}$ da $\Gamma^\mu_{\nu\alpha} = \Gamma^\mu_{\alpha\nu}$ (fra def. i a))

og $\frac{\partial^2 \tilde{x}^\mu}{\partial x^\sigma \partial x^\nu} = \underline{\Gamma^\mu_{\sigma\nu}(P)}$

Innsatt

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \left[\delta^\mu_\nu + \Gamma^\mu_{\alpha\nu}(P)(x^\alpha - x^\alpha(P))\right]\frac{\partial x^\sigma}{\partial \tilde{x}^\alpha}\frac{\partial x^\tau}{\partial \tilde{x}^\beta}\Gamma^\nu_{\sigma\tau} - \frac{\partial x^\tau}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}\Gamma^\mu_{\sigma\tau}(P)$$

i pkt P $(x^\alpha(P) - x^\alpha(P) = 0)$

$$\tilde{\Gamma}^\mu_{\alpha\beta}(P) = \frac{\partial x^\sigma}{\partial \tilde{x}^\alpha}\frac{\partial x^\tau}{\partial \tilde{x}^\beta}\Gamma^\mu_{\sigma\tau}(P) - \frac{\partial x^\tau}{\partial \tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial \tilde{x}^\beta}\Gamma^\mu_{\sigma\tau}(P) = \underline{\underline{0}}$$

sees vel i ~~bytte~~ $\sigma \leftrightarrow \tau$ i siste ledd og benytt $\Gamma^\mu_{\sigma\tau} = \Gamma^\mu_{\tau\sigma}$

Formen på bevegelseslikningen er kovariant og blir derfor i tid- og rom-punkt P i det nye koordinatsystem

$$\frac{d^2\tilde{x}^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\alpha\beta}(P)\frac{d\tilde{x}^\alpha}{d\tau}\frac{d\tilde{x}^\beta}{d\tau} = \underline{\frac{d^2\tilde{x}^\mu}{d\tau^2} = 0}$$

Som i et inertialsystem

Koordinatene $\tilde{x}^\mu$ gir et lokalt inertialsystem i P.

Løsn. Klass felt 15.1.96

2 d) Ekvivalensprinsippet: For alle legemer er den trege og den tunge massen lik $m_I = m_g$

Kraftloven gir da $m_I \vec{a} = \vec{F} = -m_g \nabla \varphi$ (φ = gravitasjonspotensial)

$$\Rightarrow \quad \vec{a} = -\nabla \varphi$$

d.v.s alle legemer akselererer likt og faller dermed like fort i et gravitasjonsfelt.

Andre formuleringer:

$\Rightarrow$ Bevegelsen for et legeme som faller i et gravitasjonsfelt hvor akselerasjonen er g, kan over et lite område og for en kort tid beskrives som fri bevegelse i en heis(rom) som akseleres oppover med akselerasjon g.

$\Rightarrow$ Gravitasjonsfeltet virkning kan lokalt og for et kort tidsrom transformeres bort ved å beskrive bevegelsen inne i en heis (et rom) som faller fritt. I et slikt lokalt inertialsystem er bevegelseslikningen $\frac{d^2 x^\mu}{d\tau^2} = 0$.

I c) har vi vist hvordan vi for et vilkårlig gravitasjonsfelt representert ved $\Gamma^\mu_{\alpha\beta}(g_{\mu\nu})$, alltid i et vilkårlig punkt P kan finne et lokalt inertialsystem hvor $\tilde{\Gamma}^\mu_{\alpha\beta}(P) = 0$ og dermed bevegelseslikningen er $\frac{d^2 \tilde{x}^\mu}{d\tau^2} = 0$ i dette punktet. I overensstemmelse med den siste formuleringen av ekvivalensprinsippet ovenfor.

Fysiske systems oppførsel i et inertialsystem bestemmes ved den spesielle relativitetsteori.

Lokale prosesser i grav. felt kan altså bestemmes ved den spesielle relativitetsteori i et lokalt inertialsystem i hvert punkt P. (Men forskjellige lokale inertialsystem for forskjellige punkt $P_1 \neq P_2$)

Løsn. Klass Felt 15.1.96

III a) $S = -mc\int ds = -mc\int d\sigma \sqrt{\left(1-\frac{\varepsilon}{r}\right)c^2\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{\varepsilon}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\vartheta}{d\sigma}\right)^2 - r^2\sin^2\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2}$

$= \int L\, d\sigma$

Plan bevegelse i ekvatorplanet $\vartheta = \frac{\pi}{2} = \text{konst}$ $d\vartheta = 0$ $\sin\vartheta = 1$

E-L likningene:

For koord. φ: $\frac{\partial L}{\partial\varphi} - \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial\sigma}\right)} = 0 \Rightarrow 0 - \frac{d}{d\sigma}\left(\frac{-r^2\left(\frac{\partial\varphi}{\partial\sigma}\right)}{\sqrt{\ }}\right) = 0 \Rightarrow \underline{\frac{d\varphi}{d\sigma} = \frac{L_0}{r^2}}$

da $\sqrt{\ } = \text{konst}$ ($L_0 = \text{konst}$)

For t: $\frac{d}{d\sigma}\left(\frac{\left(1-\frac{\varepsilon}{r}\right)c^2\frac{dt}{d\sigma}}{\sqrt{\ }}\right) = 0 \Rightarrow \underline{\underline{\frac{dt}{d\sigma} = \frac{A}{1-\frac{\varepsilon}{r}}}}$ ($A = \text{konst}$)

$\frac{dr}{d\sigma}$ finner fra intervallikningen. For lysstråler: $\frac{ds}{d\sigma} = \sqrt{\ } = 0$

$0 = \left(1-\frac{\varepsilon}{r}\right)c^2\left(\frac{dt}{d\sigma}\right)^2 - \left(1-\frac{\varepsilon}{r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\vartheta}{d\sigma}\right)^2 - r^2\sin^2\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2$

$= \left(1-\frac{\varepsilon}{r}\right)\frac{c^2A^2}{\left(1-\frac{\varepsilon}{r}\right)^2} - \frac{1}{1-\frac{\varepsilon}{r}}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{L_0}{r^2}\right)^2$ $\left(\frac{d\vartheta}{d\sigma} = 0 \quad \sin\vartheta = 1\right)$

$$\underline{\underline{\left(\frac{dr}{d\sigma}\right)^2 = c^2A^2 - \frac{L_0^2}{r^2}\left(1-\frac{\varepsilon}{r}\right)}}$$

b) ϑ-likn: $\frac{-r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2}{\sqrt{\ }} - \frac{d}{d\sigma}\left(\frac{-r^2\frac{d\vartheta}{d\sigma}}{\sqrt{\ }}\right) = 0 \Rightarrow \frac{d}{d\sigma}\left(r^2\frac{d\vartheta}{d\sigma}\right) - r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{dt}\right)^2 = 0$

eller $r^2\frac{d^2\vartheta}{d\sigma^2} + 2r\frac{dr}{d\sigma}\frac{d\vartheta}{d\sigma} - r^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0$

Hvis vi velger å sette z-aksen slik at $\vartheta = \frac{\pi}{2}$ der hvor $\frac{d\vartheta}{d\sigma} = 0$

så gir likningen: $\frac{d^2\vartheta}{d\sigma^2} = 0$ og lysstrålen forblir i $\vartheta = \frac{\pi}{2}$ planet.

c) Banelikning $r = r(\varphi)$ $\frac{dr}{d\sigma} = \frac{dr}{d\varphi}\frac{d\varphi}{d\sigma} = \frac{L_0}{r^2}\frac{dr}{d\varphi}$

Innsatt: $\left(\frac{dr}{d\sigma}\right)^2 = \left(\frac{L_0}{r^2}\frac{dr}{d\varphi}\right)^2 = c^2A^2 - \frac{L_0^2}{r^2}\left(1-\frac{\varepsilon}{r}\right) \Rightarrow \left(\frac{dr}{d\varphi}\right)^2 = \frac{c^2A^2}{L^2}r^4 - r^2 + \varepsilon r$

$$\underline{\underline{\left(\frac{dr}{d\varphi}\right)^2 = \varepsilon r - r^2 + \frac{r^4}{b^2}}} \quad \text{med} \quad b^2 \equiv \frac{L_0^2}{c^2A^2}$$

d) For sirkel bane $r = \frac{3}{2}\varepsilon \equiv r_c$ må $\frac{dr}{d\varphi} = 0$

Setter inn: $r\left(\varepsilon - \frac{3}{2}\varepsilon + \frac{1}{b^2}\left(\frac{3}{2}\varepsilon\right)^3\right) = r\varepsilon\left(-\frac{1}{2} + \frac{27}{8}\frac{\varepsilon^2}{b^2}\right) = 0$ hvis $\underline{\underline{b^2 = \frac{27}{4}\varepsilon^2}}$

Undersøker stabiliteten ved å perturbere banen litt $r = r_B + \delta$ og se om δ vokser eller minker videre:

$\frac{d^2(r_B+\delta)}{d\varphi^2} = \frac{d^2r_B}{d\varphi^2} + \frac{d^2\delta}{d\varphi^2} = \frac{1}{2}\varepsilon - (r_B+\delta) + \frac{2}{b^2}(r_B+\delta)^3 = \frac{1}{2}\varepsilon - r_B + \frac{2}{b^2}r_B^3 - \delta + \frac{2}{b^2}\left(3r_B^2\delta + 3r_B\delta^2 + \delta^3\right)$

$\frac{d^2\delta}{d\varphi^2} = \delta\left(-1 + \frac{6}{b^2}r_B^2\right) + \frac{6r_B}{b^2}\delta^2 + \frac{2}{b^2}\delta^3 = \underline{\delta + \frac{4}{3\varepsilon}\delta^2 + \frac{8}{27\varepsilon^2}\delta^3}$ med $r_B = \frac{3}{2}\varepsilon$ og $b^2 = \frac{27}{4}\varepsilon^2$

Positiv $\delta > 0$ gir $\frac{d^2\delta}{d\varphi^2} > 0$ d.v.s $\delta(\varphi)$ krummer oppover. På sirkelen er $\frac{dr}{d\varphi} = 0$ så δ vokser.

Negativ $\delta < 0$ gir $\frac{d^2\delta}{d\varphi^2} < 0$ krumming nedover og $|\delta|$ øker. D.v.s <u>ustabil bane</u>.