Klassisk felt teori. Kontinuasjonseksamen 28.08.96

1a Hamiltons variasjonsprinsipp

$$\delta S = \delta \int \mathcal{L}\, d^4x = \int \left[\frac{\partial \mathcal{L}}{\partial \varphi_n}\delta\varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)}\delta(\partial_\mu\varphi_n)\right] d^4x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \varphi_n} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)}\right]\delta\varphi_n\, d^4x + \underbrace{\int_{\text{overflate}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_n)}\delta\varphi_n\, dS^\mu}_{0} = 0$$

0 når $\delta\varphi_n$ er fri bare med $\delta\varphi_n = 0$ på overflaten

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \varphi_n} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)} = 0 \qquad n = 1,2\dots N$$

1b

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_n)}\partial^\nu\varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_n)}\partial_\mu\partial^\nu\varphi_n - g^{\mu\nu}\frac{d\mathcal{L}}{dx^\mu}$$

$$= \frac{\partial \mathcal{L}}{\partial \varphi_n}\partial^\nu\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi_n)}\partial_\mu\partial^\nu\varphi_n - g^{\mu\nu}\left(\frac{\partial \mathcal{L}}{\partial\varphi_n}\partial_\mu\varphi_n + \frac{\partial \mathcal{L}}{\partial(\partial_\nu\varphi_n)}\partial_\mu\partial^\nu\varphi_n + \frac{\partial \mathcal{L}}{\partial x^\mu}\right) = 0$$

når $\mathcal{L}$ ikke avhenger eksplisitt av x^μ : $\frac{\partial \mathcal{L}}{\partial x^\mu} = 0$ som her.

1c Dreieimpulstettheten $\mathcal{M}^0{}_i$ er en av komponentene

$$\mathcal{M}^\mu{}_i = \frac{1}{c}\varepsilon_{ijk}\, x^j\, T^{\mu k} \qquad \mu = 0,1,2,3$$

Lokal bevarelsessetning

$$\frac{\partial \mathcal{M}^\mu{}_i}{\partial x^\mu} = \frac{1}{c}\varepsilon_{ijk}\,\delta^j{}_\mu\, T^{\mu k} + \frac{1}{c}\varepsilon_{ijk}\, x^j \underbrace{\frac{\partial T^{\mu k}}{\partial x^\mu}}_{0}$$

0 når impulsbevarelse som her

$$= \frac{1}{c}\frac{1}{2}\left(\varepsilon_{ijk}T^{jk} + \varepsilon_{ikj}T^{kj}\right) \quad \text{(Har skiftet navn på sumindekser i siste ledd)}$$

$$= \frac{1}{2c}\varepsilon_{ijk}\left(T^{jk} - T^{kj}\right) = 0 \quad \text{når } T^{jk} = T^{kj}$$

Utskrevet:

$$\frac{\partial \mathcal{M}^\mu{}_i}{\partial x^\mu} = \frac{1}{c}\frac{\partial}{\partial t}\left(x^j\frac{1}{c}T^{0k} - x^k\frac{1}{c}T^{0j}\right) - \partial_\ell\left(x^j\frac{1}{c}T^{\ell k} - x^k\frac{1}{c}T^{kj}\right) = 0$$

$$= \frac{1}{c}\left[\frac{\partial}{\partial t}\left(x^j g^k - x^k g^j\right) - \frac{\partial}{\partial x^\ell}\left(x^j\sigma^{\ell k} - x^k\sigma^{\ell j}\right)\right] = \frac{\partial}{\partial t}\mathcal{M}^0{}_i + \frac{\partial}{\partial x^\ell}\mathcal{D}^\ell{}_i = 0$$

2a Varierer banen $x^\mu \to x^\mu + \delta x^\mu$ med faste endepkt.

$$\delta S = \int\left[\frac{\partial L}{\partial x^k}\delta x^k + \frac{\partial L}{\partial \dot{x}^k}\delta\dot{x}^k\right]dt = \int\left(\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k}\right)\delta x^k\, dt = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} = 0$$

2b

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^k}\dot{q}^k\right) - \frac{dL}{dt} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}\right)\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k - \frac{\partial L}{\partial q^k}\dot{q}^k - \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k - \frac{\partial L}{\partial t} \quad \text{Bruker så feltl.lign.}$$

$$= \frac{\partial L}{\partial q^k}\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k - \frac{\partial L}{\partial q^k}\dot{q}^k - \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} = 0 \quad \text{hvis } L \text{ ikke inneholder } t \text{ eksplisitt}$$

2c

$$E = \frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L = \frac{1}{2}(a_{ki} + a_{ik})\dot{q}^i\dot{q}^k + b_i\dot{q}^i - \frac{1}{2}a_{ik}\dot{q}^i\dot{q}^k - b_i\dot{q}^i + V(q)$$

$$= \frac{1}{2}a_{ki}\dot{q}^i\dot{q}^k + V(q)$$

Leddet $b_i\dot{q}^i$ bidrar ikke til energiinnholdet.
Energien er symmetrisk i i og k selv om $a_{ik} \neq a_{ki}$.

Klas. felt 28.08.96 forts.

2d $\delta\int_1^2 ds = 0$ Baneparameter σ: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}d\sigma^2 \equiv w^2 d\sigma^2$

$\delta\int_1^2 w\, d\sigma = \int \delta w\, d\sigma = 0$

$\delta w = \frac{1}{2w}\delta(w^2) = \frac{1}{2w}\left[\delta g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} + 2g_{\mu\nu}\frac{dx^\mu}{d\sigma}\delta\left(\frac{dx^\nu}{d\sigma}\right)\right]$

Partiell integrasjon uten variasjon på grensene gir ($\delta g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda$)

$\delta\int_1^2 ds = \int_1^2 \left[\frac{1}{2w}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\delta x^\lambda - \frac{d}{d\sigma}\left(\frac{g_{\mu\nu}}{w}\frac{dx^\mu}{d\sigma}\right)\delta x^\nu\right] d\sigma = 0$

Med egentiden som parameter (forutsatt: ikke lyslike hvor $ds = 0$)

$d\sigma = c\,d\tau \quad w = 1$

$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$

$g_{\mu\lambda}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$

$g_{\mu\lambda}\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\lambda\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$ Multipliser med $g^{\kappa\lambda}$ og $\sum_\lambda$ $\quad (g_{\mu\lambda}g^{\kappa\lambda} = \delta_\mu^\kappa)$

$\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \Rightarrow \frac{dV^\kappa}{d\tau} + \Gamma^\kappa_{\mu\nu}V^\mu V^\nu = 0$

3a Med gitt metrikk

$S = -mc\int ds = -mc\int d\sigma\sqrt{\frac{r-A}{r+A}c^2\left(\frac{dt}{d\sigma}\right)^2 - \frac{r+A}{r-A}\left(\frac{dr}{d\sigma}\right)^2 - (r+A)^2\left(\left(\frac{d\vartheta}{d\sigma}\right)^2 + \sin^2\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2\right)}$

Når $\vartheta = \frac{\pi}{2}$ blir siste ledd $(r+A)^2\left(\frac{d\varphi}{d\sigma}\right)^2$

Variasjon $\frac{\delta S}{\delta\varphi(\sigma)} = 0$ gir bevegelseslikn. $\frac{\partial L}{\partial\varphi} - \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{d\varphi}{d\sigma}\right)} = 0 \Rightarrow \frac{d}{d\sigma}\left(\frac{(r+A)^2\frac{d\varphi}{d\sigma}}{\sqrt{\ }}\right) = 0$

$\Rightarrow \frac{d\varphi}{d\sigma} = \frac{L_0}{(r+A)^2}$ $\quad L_0$ = konst

$\frac{\delta S}{\delta t} = 0 \Rightarrow \frac{d}{d\sigma}\left[\frac{\frac{r-A}{r+A}c^2\frac{dt}{d\sigma}}{\sqrt{\ }}\right] = 0 \Rightarrow \frac{dt}{d\sigma} = \frac{B}{\frac{r-A}{r+A}}$ $\quad B$ = konst

Bevegelseslikn. for $\frac{dr}{d\sigma}$ finnes enklest fra intervallikn.

$\frac{r-A}{r+A}\frac{c^2B^2}{\left(\frac{r-A}{r+A}\right)^2} - \frac{r+A}{r-A}\left(\frac{dr}{d\sigma}\right)^2 - (r+A)^2\left(\frac{L_0}{(r+A)^2}\right)^2 = 0$

$\left(\frac{dr}{d\sigma}\right)^2 = c^2B^2 - \frac{L_0^2(r-A)}{(r+A)^3} = c^2B^2 - \frac{L_0^2}{(r+A)^2}\frac{r-A}{r+A}$

3b $\frac{\partial L}{\partial\vartheta} - \frac{d}{d\sigma}\frac{\partial L}{\partial\left(\frac{d\vartheta}{d\sigma}\right)} = 0 \Rightarrow -\frac{(r+A)^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2}{\sqrt{\ }} - \frac{d}{d\sigma}\left(\frac{-(r+A)^2\frac{d\vartheta}{d\sigma}}{\sqrt{\ }}\right) = 0$

$\Rightarrow (r+A)^2\frac{d^2\vartheta}{d\sigma^2} + 2(r+A)\frac{dr}{d\sigma}\frac{d\vartheta}{d\sigma} - (r+A)^2\sin\vartheta\cos\vartheta\left(\frac{d\varphi}{d\sigma}\right)^2 = 0$

Velger z-akse slik at $\vartheta = \frac{\pi}{2}$ der hvor $\frac{d\vartheta}{d\sigma} = 0$ da blir $\frac{d^2\vartheta}{d\sigma^2} = 0$ der og lyset forblir i $\frac{\pi}{2}$-planet.

3c Finner $r = r(\varphi)$

$$\frac{dr}{d\tau} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = \frac{dr}{d\varphi}\frac{L_0}{(r+A)^2}$$

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{L_0}{(r+A)^2}\frac{dr}{d\varphi}\right)^2 = c^2B^2 - \frac{L_0^2}{(r+A)^2}\frac{r-A}{r+A} \Rightarrow \left(\frac{dr}{d\varphi}\right)^2 = \frac{c^2B^2(r+A)^4}{L_0^2} - (r-A)(r+A)$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{(r+A)^4}{b^2} + A^2 - r^2 \qquad \text{med} \qquad b^2 = \frac{L_0^2}{c^2B^2}$$

3d Sirkel $r = 2A$ $\quad \frac{dr}{d\varphi} = 0$

$$0 = A^2 - 4A^2 + \frac{(3A)^4}{b^2} \qquad b^2 = \frac{3A^4}{3A^2} = \underline{\underline{27A^2}} \qquad \underline{b = 3\sqrt{3}\,A}$$

3e

$$ds^2 = \left(1 - \frac{2A}{r+A}\right)c^2dt^2 - \frac{1}{1-\frac{2A}{r+A}}\left(d(r+A)\right)^2 - (r+A)^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

$$= \left(1 - \frac{2A}{\rho}\right)c^2dt^2 - \frac{1}{1-\frac{2A}{\rho}}d\rho^2 - \rho^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \qquad \text{med } \rho = r+A$$

Som er Schw. metrikken med $\underline{2A = \varepsilon = \frac{2GM}{c^2} \qquad A = \frac{\varepsilon}{2}}$

r betyr radiell avstand fra en kuleflate med radius $A = \frac{\varepsilon}{2}$ fra origo. r regnet fra $\frac{1}{2}$ Schw. radius,