Solution to
Exam 4. december 2010
FY2045/TFY4250 Quantum Mechanics I

Problem 1

a. ♠A bound state for this potential must have energy $E < 0$. For $x \neq \pm a$, the time-independent Schrödinger equation then takes the form

$$\psi_E'' = \frac{2m}{\hbar^2}[V(x) - E]\psi_E = \underbrace{-\frac{2mE}{\hbar^2}}_{>0}\,\Psi_E \equiv \kappa^2\psi_E\ , \quad \text{with} \quad \kappa \equiv \frac{1}{\hbar}\sqrt{2m(-E)}.$$

The general solution for the region $x > a$ has the form

$$\Psi_E = C\,e^{-\kappa x} + D\,e^{\kappa x}.$$

Here, we must set $D = 0$ in order to have a solution that does not diverge when $x \to \infty$. This is what we wanted to show.

♠For $-a < x < a$, a symmetric energy eigenfunction must be a symmetric combination of $e^{\kappa x}$ and $e^{-\kappa x}$, and this is precisely the given form

$$\psi_E = A\cosh\kappa x = \frac{A}{2}(e^{\kappa x} + e^{-\kappa x}), \quad \text{q.e.d.}$$

♠For $x < -a$, the energy eigenfunction must have the form $C'e^{\kappa x}$. This implies that the energy eigenfunction does not have any node in the regions $x < -a$ and $x > a$. Any possible node must therefor occur *between* the two wells. Since the above solution for $-a < x < a$ is also without nodes, it follows that the only symmetric bound state is the ground state, without nodes. If a next symmetric bound state were to exist, it should have two nodes (for $-a < x < a$) and that, as we see, is impossible.

b. ♠An antisymmetric bound state must for $-a < x < a$ be an *antisymmetric* linear combination of $e^{\kappa x}$ and $e^{-\kappa x}$:

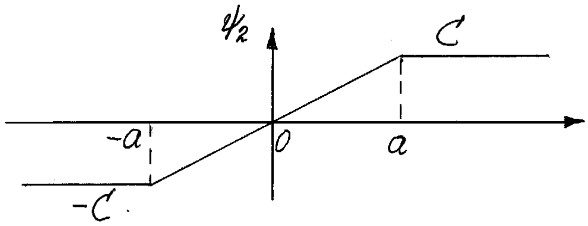
$$\psi = \frac{B}{2}(e^{\kappa x} - e^{-\kappa x}) = B\sinh\kappa x.$$

Then, ψ/B is positive for $0 < x < a$ and negative for $-a < x < 0$, and thus in the region between the wells has only the single node at the origin . From the above discussion we then realize that an antisymmetric bound state can have only one zero. This means that we can at most have two bound energy eigenfunctions, one symmetric, and possibly one antisymmetric.

♠For $\beta = \beta_0$, the wells are just too "weak" to make the first excited state a *bound* state. This means that the energy of the first excited state is in this case equal to zero. From **a** we then have

$$\psi'' = -\frac{2mE}{\hbar^2}\,\psi = 0$$

outside the delta-function wells, so that ψ becomes linear everywhere, except for $x = \pm a$, where it has kinks. Because the solution is not allowed to become infinite (for $x \to \pm\infty$), it follows that it must look like this:



With $\psi(a) = C$, $\psi'(a^+) = 0$ and $\psi'(a^-) = C/a$, it follows from the given discontinuity condition that

$$0 - \frac{C}{a} = -\frac{2m\beta_0}{\hbar^2}\, C \qquad \Longleftrightarrow \qquad \beta_0 = \frac{\hbar^2}{2ma}.$$

For $\beta > \beta_0$, we have a bound antisymmetric state with energy $E_2 < 0$. This energy eigenfunction will curve away from the axis (except in the points $x = \pm a$). For $\beta \le \beta_0$, only the ground state is bound, while the first excited state is as sketched above. (The energy spectrum is continuous for $E \ge 0$.)

Problem 2

a. ♠Suppose that the piston is at the position a. The ground state for particle 1 then has the form

$$\psi_1 = A \sin k_x x \ \sin k_y y \ \sin k_z z,$$

where $k_x a = \pi$, $k_y L_y = \pi$ og $k_z L_z = \pi$. Particle 1 then has the energy

$$E_1 = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{a^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}\right).$$

In the same manner we find that the particle in chamber 2 has the energy

$$E_2 = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{(L_x - a)^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}\right),$$

so that the total energy of the two particles is

$$E(a) = E_1 + E_2 = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{a^2} + \frac{1}{(L_x - a)^2} + \frac{2}{L_y^2} + \frac{2}{L_z^2}\right).$$

This energy is minimal when

$$\frac{d}{da}\left(\frac{1}{a^2} + \frac{1}{(L_x - a)^2}\right) = 2\left[-\frac{1}{a^3} + \frac{1}{(L_x - a)^3}\right] = 0,$$

which is satisfied for $a = L_x - a$, that is for

$$a = a_0 = L_x/2.$$

[It is easily seen that the second derivative here is positive. Thus the energy really has a minimum, just as we expect intuitively.]

b. ♠If we *imagine* that the piston is moved an infinitesimal distance da, the external force F_x will do a work which leads to a net energy increase $dE = F_x da$ for the two particles. Thus (from the calculation in i **a**),

$$F_x(a) = \frac{dE(a)}{da} = \frac{\hbar^2\pi^2}{2m}\cdot 2\left[\frac{1}{(L_x-a)^3} - \frac{1}{a^3}\right].$$

At the equilibrium position, $a = a_0 = L_x/2$, this force is of course equal to zero.

♠For $a = a_1 = L_x/3$, the force is

$$F_x(L_x/3) = \frac{\hbar^2\pi^2}{m}\left[\frac{1}{(L_x - L_x/3)^3} - \frac{1}{(L_x/3)^3}\right] = -\frac{7\cdot 27}{8}\frac{\hbar^2\pi^2}{mL_x^3}.$$

The sign tells tells us that this force point to the left; when chamber 1 is only half as long as chamber 2, the quantum pressure is highest in chamber 1.

♠With 8 bosons in chamber 1 and only one in chamber 2, the energy in chamber 1 becomes 8 times larger than above, since all 8 bosons can be in the lowest one-particle state, even if they are identical. Thus this system has a total energy

$$E(a) = 8E_1 + E_2 = \frac{\hbar^2\pi^2}{2m}\left[\frac{8}{a^2} + \frac{1}{(L_x-a)^2} + 16(1/L_y^2 + 1/L_z^2)\right].$$

This is minimal when

$$\frac{d}{da}\left[\frac{8}{a^2} + \frac{1}{(L_x-a)^2}\right] = 2\left[\frac{1}{(L_x-a)^3} - \frac{8}{a^3}\right] = 0,$$

which is satisfied when $a^3 = 8(L_x-a)^3$, that is, when $a = 2(L_x - a)$, that is, for

$$a = a_2 = 2L_x/3.$$

Problem 3

a. ♠With

$$\chi(0) = \begin{pmatrix}\cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta\end{pmatrix} \equiv \begin{pmatrix} a_0 \\ b_0\end{pmatrix}$$

we have that $2a_0^* b_0 = \sin\theta$ and $|a_0|^2 - |b_0|^2 = \cos\theta$. The spin direction immediately after the measurement then is

$$\langle\,\boldsymbol{\sigma}\,\rangle_0 = \hat{\mathbf{x}}\,\Re\mathrm{e}(2a_0^*b_0) + \hat{\mathbf{y}}\,\Im\mathrm{m}(2a_0^*b_0) + \hat{\mathbf{z}}\,(|a_0|^2 - |b_0|^2) = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta.$$

♠We calculate

$$\begin{aligned}\mathbf{S}\cdot\langle\,\boldsymbol{\sigma}\,\rangle_0\,\chi(0) &= \tfrac{1}{2}\hbar\boldsymbol{\sigma}\cdot(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\,\chi(0) = \tfrac{1}{2}\hbar\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}\begin{pmatrix}\cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta\end{pmatrix} \\ &= \tfrac{1}{2}\hbar\begin{pmatrix}\cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta\end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix}\cos(\theta - \frac{1}{2}\theta) \\ \sin(\theta-\frac{1}{2}\theta)\end{pmatrix} = \tfrac{1}{2}\hbar\,\chi(0).\end{aligned}$$

Thus the initial state $\chi(0)$ is an eigenstate of $\mathbf{S}\cdot\langle\,\boldsymbol{\sigma}\,\rangle_0$ with the eigenvalue $\frac{1}{2}\hbar$.

♠According to the measurement postulate, the measurement of $\mathbf{S}\cdot\hat{\mathbf{n}}$ must give one of the eigenvalues $\pm\frac{1}{2}\hbar$ and leave the spin in the corresponding eigenstate. We can then conclude that the measurement direction was $\hat{\mathbf{n}} = \langle\,\boldsymbol{\sigma}\,\rangle_0 = \hat{\mathbf{x}}\,\sin\theta + \hat{\mathbf{z}}\,\cos\theta$, and that the measured result was $\mathbf{S}\cdot\hat{\mathbf{n}} = +\frac{1}{2}\hbar$.

b. ♠The state of the spin for $t > 0$ must be a linear combination of the two stationary states of this system:

$$\chi(t) = c_+\chi_+ e^{-i\omega t/2} + c_-\chi_- e^{i\omega t/2} = \begin{pmatrix} c_+ e^{-i\omega t/2} \\ c_- e^{i\omega t/2} \end{pmatrix}.$$

Here we have used the energy eigenvalues $E_\pm = \omega S_z = \pm\frac{1}{2}\hbar\omega$. For $t = 0$ we then have

$$\chi(0) = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad \Longrightarrow \qquad c_+ = \cos\tfrac{1}{2}\theta, \quad c_- = \sin\tfrac{1}{2}\theta,$$

so that the state for $t \ge 0$ is

$$\chi(t) = \begin{pmatrix} \cos\frac{1}{2}\theta\, e^{-i\omega t/2} \\ \sin\frac{1}{2}\theta\, e^{i\omega t/2} \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix}.$$

With $2a^*b = \sin\theta e^{i\omega t}$ and $|a|^2 - |b|^2 = \cos\theta$, the spin direction at time t then is

$$\begin{aligned} \langle\, \boldsymbol{\sigma}\, \rangle_t &= \hat{\mathbf{x}}\, \sin\theta\, \cos\omega t + \hat{\mathbf{y}}\, \sin\theta\, \sin\omega t + \hat{\mathbf{z}}\, \cos\theta \\ &= (\hat{\mathbf{x}}\, \cos\omega t + \hat{\mathbf{y}}\, \sin\omega t)\, \sin\theta + \hat{\mathbf{z}}\, \cos\theta. \end{aligned}$$

Here, we see that the spin direction precesses around the direction $\hat{\mathbf{z}}$ of $\mathbf{B}$ with the angular frquency ω.

c. ♠From the formula for $\chi(t)$ we see that $\chi(2\pi/\omega) = -\chi(0)$. This change of sign of the spinor gives exactly the same $\langle\, \boldsymbol{\sigma}\, \rangle = \chi^\dagger \boldsymbol{\sigma} \chi$ as at $t = 0$, in agreement with the formula for $\langle\, \boldsymbol{\sigma}\, \rangle_t$.

♠We remember that the measurement of $\mathbf{S}\cdot\hat{\mathbf{n}} = \mathbf{S}\cdot(\hat{\mathbf{x}}\, \sin\theta + \hat{\mathbf{z}}\cos\theta) = \frac{1}{2}\hbar$ leaves the system in the state

$$\chi(0) = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}.$$

In analogy, the measurement of $\mathbf{S}\cdot\hat{\mathbf{n}}' = \mathbf{S}\cdot(\hat{\mathbf{x}}\, \sin\theta' + \hat{\mathbf{z}}\cos\theta') = \frac{1}{2}\hbar$ will leave the system in the state

$$\chi_{\text{after}} = \begin{pmatrix} \cos\frac{1}{2}\theta' \\ \sin\frac{1}{2}\theta' \end{pmatrix}.$$

♠The probability *amplitude* of the latter result is the projection of the state before the measurement onto the state after the measurement, that is,

$$A = \chi^\dagger_{\text{after}}\chi_{\text{before}} = (\cos\tfrac{1}{2}\theta' \;\; \sin\tfrac{1}{2}\theta') \begin{pmatrix} -\cos\frac{1}{2}\theta \\ -\sin\frac{1}{2}\theta \end{pmatrix} = -\cos(\tfrac{1}{2}\theta' - \tfrac{1}{2}\theta).$$

Thus the probability is

$$P = |A|^2 = \cos^2(\tfrac{1}{2}\theta' - \tfrac{1}{2}\theta) = \tfrac{1}{2}[1 + \cos(\theta' - \theta)] = \tfrac{1}{2}[1 + \hat{\mathbf{n}}'\cdot\hat{\mathbf{n}}].$$

Problem 4

a. ♠By using the eigenvalue equation and its adjoint, $\langle\Psi|a_x^\dagger = \alpha^*\langle\Psi|$, we find that the expectation value of x in the state $|\Psi\rangle$ is

$$\langle\, x\,\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\Psi|(a_x^\dagger + a_x)|\Psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\alpha^* + \alpha), \quad \text{q.e.d.}$$

Similarly, we find that

$$\langle\, p_x\,\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle\Psi|(a_x^\dagger - a_x)|\Psi\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,(\alpha^* - \alpha), \quad \text{q.e.d.}$$

♠Using the commutator relation $a_x a_x^\dagger = 1 + a_x^\dagger a_x$, we find that

$$\widehat{x}^2 = \frac{\hbar}{2m\omega}(a_x^\dagger + a_x)(a_x^\dagger + a_x) = \frac{\hbar}{2m\omega}(a_x^\dagger a_x^\dagger + a_x a_x + 2a_x^\dagger a_x + 1),$$

and similarly that

$$\widehat{p}_x^2 = -\frac{m\hbar\omega}{2}(a_x^\dagger a_x^\dagger + a_x a_x - 2a_x^\dagger a_x - 1).$$

This gives

$$\left\langle\, x^2\,\right\rangle = \frac{\hbar}{2m\omega}[(\alpha^*)^2 + \alpha^2 + 2\alpha^*\alpha + 1] = \frac{\hbar}{2m\omega}[(\alpha^* + \alpha)^2 + 1]$$

and

$$\left\langle\, \widehat{p}_x^2\,\right\rangle = \frac{m\hbar\omega}{2}[1 - (\alpha^* - \alpha)^2].$$

Using these results, we find the uncertainties

$$\Delta x = \sqrt{\langle\, x^2\,\rangle - \langle\, x\,\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \qquad \text{and} \qquad \Delta p_x = \sqrt{\frac{m\hbar\omega}{2}},$$

which give $\Delta x \Delta p_x = \frac{1}{2}\hbar$.

b. ♠The probability density of the initial state,

$$|\Psi_b(x,y,0)|^2 = C_0^4 e^{-m\omega(x-b)^2/\hbar} e^{-m\omega y^2/\hbar},$$

is symmetric with respect to the line $x = b$ and also with respect to the line $y = 0$. This means that

$$\langle\, x\,\rangle_0 = b \qquad \text{and} \qquad \langle\, y\,\rangle_0 = 0.$$

♠Calculating

$$\begin{aligned}
a_x^{\rm pr}\Psi_b(x,y,0) &= \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Psi_b(x,y,0) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}(m\omega(x-b)/\hbar)\right)\Psi_b(x,y,0) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\, b\,\Psi_b(x,y,0),
\end{aligned}$$

we see that the initial state is indeed an eigenstate of $a_x^{\rm pr}$, with the eigenvalue

$$\alpha_x(0) = \sqrt{\frac{m\omega}{2\hbar}}\, b.$$

In the same manner we find that

$$\begin{aligned} a_y^{\rm pr}\Psi_b(x,y,0) &= \sqrt{\frac{m\omega}{2\hbar}}\left(y + \frac{\hbar}{m\omega}\frac{\partial}{\partial y}\right)\Psi_b(x,y,0) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(y + \frac{\hbar}{m\omega}(-m\omega y/\hbar + im\omega b/\hbar)\right)\Psi_b(x,y,0) \\ &= i\sqrt{\frac{m\omega}{2\hbar}}\, b\,\Psi_b(x,y,0). \end{aligned}$$

Thus the igenvalue of $a_y^{\rm pr}$ is

$$\alpha_y(0) = i\sqrt{\frac{m\omega}{2\hbar}}\, b = i\alpha_x(0), \quad \text{q.e.d.}$$

<u>**c.**</u> ♠From the above results, it follows that

$$\langle\, x\,\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha_x(t) + \alpha_x^*(t)\right] = \sqrt{\frac{\hbar}{2m\omega}}\cdot 2\Re\mathrm{e}\left(\alpha_x(0)\, e^{-i\omega t}\right) = b\cos\omega t$$

and

$$\langle\, y\,\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha_y(t) + \alpha_y^*(t)\right] = \sqrt{\frac{\hbar}{2m\omega}}\cdot 2\Re\mathrm{e}\left(i\alpha_x(0)\, e^{-i\omega t}\right) = b\sin\omega t.$$

As a check, we note that the values for $t = 0$ agree with the results in <u>**b**</u>.

♠Since

$$\langle\, x\,\rangle_t^2 + \langle\, y\,\rangle_t^2 = b^2,$$

we can state that the expectation value of the position is moving in a circular orbit with radius b, and with angular velocity ω.

♠We start by noting that the product of the two one-dimensional solutions reproduces the initial state specified above. Then it only remains to show that this product satisfies the time-dependent Schrödinger equation: Inserting, we find that

$$\left[i\hbar\frac{\partial}{\partial t} - \widehat{H}\right]\Psi_x(x,t)\Psi_y(y,t) = \left[i\hbar\frac{\partial}{\partial t} - \widehat{H}^{(x)} - \widehat{H}^{(y)}\right]\Psi_x(x,t)\Psi_y(y,t)$$

$$\begin{aligned} &= \left[i\hbar\frac{\partial}{\partial t}\,\Psi_x(x,t)\right]\Psi_y(y,t) + \Psi_x(x,t)i\hbar\frac{\partial}{\partial t}\,\Psi_y(y,t) \\ &\quad - \left[\widehat{H}^{(x)}\Psi_x(x,t)\right]\Psi_y(y,t) - \Psi_x(x,t)\,\widehat{H}^{(y)}\Psi_y(y,t) \\ &= \left[\left(i\hbar\frac{\partial}{\partial t} - \widehat{H}^{(x)}\right)\Psi_x(x,t)\right]\Psi_y(y,t) + \Psi_x(x,t)\left(i\hbar\frac{\partial}{\partial t} - \widehat{H}^{(y)}\right)\Psi_y(y,t) \\ &= 0, \quad \text{q.e.d.} \end{aligned}$$

Løsningsforslag
Eksamen 4. desember 2010
FY2045/TFY4250 Kvantemekanikk I

Oppgave 1

a. ♠En bunden tilstand i det aktuelle potensialet må ha energi $E < 0$. For $x \neq \pm a$ tar den tidsuavhengige Schrödingerligningen da formen

$$\psi_E'' = \frac{2m}{\hbar^2}[V(x) - E]\psi_E = \underbrace{-\frac{2mE}{\hbar^2}}_{>0}\,\Psi_E \equiv \kappa^2\psi_E\;, \quad \text{med} \quad \kappa \equiv \frac{1}{\hbar}\sqrt{2m(-E)}.$$

Den generelle løsningen for området $x > a$ har formen

$$\Psi_E = C\,e^{-\kappa x} + D\,e^{\kappa x}.$$

Her må vi sette $D = 0$ for å få en løsning som ikke går mot uendelig når $x \to \infty$. Det var dette vi skulle vise.

♠For $-a < x < a$ må en symmetrisk energiegenfunksjon være en symmetrisk kombinasjon av $e^{\kappa x}$ og $e^{-\kappa x}$, og dette er nettopp

$$\psi_E = A\cosh\kappa x = \frac{A}{2}(e^{\kappa x} + e^{-\kappa x}), \quad \text{q.e.d.}$$

♠For $x < -a$ må energiegenfunksjonen ha formen $C'e^{\kappa x}$. Dette innebærer at energiegenfunksjonen er fri for nullpunkter både for $x < -a$ og for $x > a$. Eventuelle nullpunkter må derfor ligge *mellom* de to brønnene. Da løsningen ovenfor for $-a < x < a$ er fri for nullpunkter, følger det at den eneste symmetriske bundne tilstanden er grunntilstanden, uten nullpunkter. Dersom det skulle eksistere en neste symmetrisk bunden tilstand (en 2. eksiterte), måtte denne ha to nullpunkter (for $-a < x < a$) og det er altså ikke mulig.

b. ♠En antisymmetrisk bunden tilstand må for $-a < x < a$ være en *antisymmetrisk* lineærkombinasjon av $e^{\kappa x}$ og $e^{-\kappa x}$:

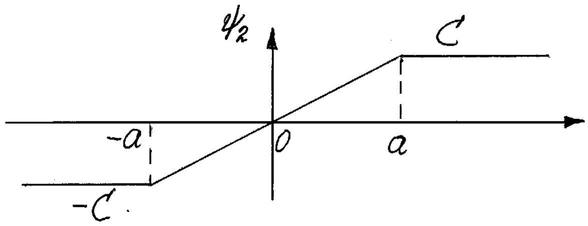
$$\psi = \frac{B}{2}(e^{\kappa x} - e^{-\kappa x}) = B\sinh\kappa x.$$

Da er ψ/B positiv for $0 < x < a$ og negativ for $-a < x < 0$, og har følgelig i dette området mellom brønnene bare det ene nullpunktet i origo. Fra diskusjonen ovenfor innser vi da at dette er det eneste nullpunktet en antisymmetrisk bunden tilstand kan ha. Vi kan altså maksimalt ha to bundne energiegenfunksjoner, én symmetrisk og muligens én antisymmetrisk. .

♠For $\beta = \beta_0$ er brønnene akkurat for svake til å gi binding for 1. eksiterte tilstand, dvs at energien for denne tilstanden er lik null. Fra pkt. **a** har vi da at

$$\psi'' = -\frac{2mE}{\hbar^2}\,\psi = 0$$

utenfor delta-brønnene, slik at ψ må være lineær over alt, unntatt for $x = \pm a$, hvor den har knekkpunkter. Løsningen ser da slik ut (idet ψ ikke får lov til å gå mot uendelig for $x \to \pm\infty$):



Med $\psi(a) = C$, $\psi'(a^+) = 0$ og $\psi'(a^-) = C/a$ har vi fra den oppgitte diskontinuitetsbetingelsen:

$$0 - \frac{C}{a} = -\frac{2m\beta_0}{\hbar^2}C \qquad \Longleftrightarrow \qquad \beta_0 = \frac{\hbar^2}{2ma}.$$

For $\beta > \beta_0$ har vi en bunden antisymmetrisk tilstand med energi $E_2 < 0$. Denne vil krumme bort fra aksen (unntatt i punktene $x = \pm a$). For $\beta \le \beta_0$ er bare grunntilstanden bunden, mens første eksiterte tilstand er som skissert ovenfor. (Husk at energispektret er kontinuerlig for $E \ge 0$.)

Oppgave 2

a. ♠Anta at stempelet er i posisjonen a. I grunntilstanden har energiegenfunksjonen for partikkelen i kammer 1 da formen

$$\psi_1 = A \sin k_x x \; \sin k_y y \; \sin k_z z,$$

der $k_x a = \pi$, $k_y L_y = \pi$ og $k_z L_z = \pi$. Partikkel 1 har da energien

$$E_1 = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2) = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{a^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}\right).$$

På tilsvarende måte finner vi at partikkelen i kammer 2 har energien

$$E_2 = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{(L_x - a)^2} + \frac{1}{L_y^2} + \frac{1}{L_z^2}\right),$$

slik at den totale energien til de to partiklene er

$$E(a) = E_1 + E_2 = \frac{\hbar^2\pi^2}{2m}\left(\frac{1}{a^2} + \frac{1}{(L_x - a)^2} + \frac{2}{L_y^2} + \frac{2}{L_z^2}\right).$$

Denne er minimal når

$$\frac{d}{da}\left(\frac{1}{a^2} + \frac{1}{(L_x - a)^2}\right) = 2\left[-\frac{1}{a^3} + \frac{1}{(L_x - a)^3}\right] = 0,$$

som er oppfylt for $a = L_x - a$, dvs for

$$a = a_0 = L_x/2.$$

[Det er lett å sjekke at den andrederiverte her er positiv, slik at energien virkelig har et minimum, noe vi vel senser intuitivt.]

b. ♠Dersom vi *tenker oss* at stempelet beveges et stykke da, vil den ytre kraften F_x utføre et arbeid som svarer til en netto energiøkning $dE = F_x da$ for de to partiklene. Vi har altså (fra utregningen i pkt. **a**)

$$F_x(a) = \frac{dE(a)}{da} = \frac{\hbar^2\pi^2}{2m} \cdot 2\left[\frac{1}{(L_x - a)^3} - \frac{1}{a^3}\right].$$

I likevektsposisjonen $a = a_0 = L_x/2$ er denne kraften selvsagt lik null.

♠For $a = a_1 = L_x/3$ er den

$$F_x(L_x/3) = \frac{\hbar^2\pi^2}{m}\left[\frac{1}{(L_x - L_x/3)^3} - \frac{1}{(L_x/3)^3}\right] = -\frac{7\cdot 27}{8}\frac{\hbar^2\pi^2}{mL_x^3}.$$

Fortegnet forteller at kraften peker mot venstre; når kammer 1 er bare halvparten så langt som kammer 2, er kvantetrykket størst i kammer 1.

♠Med 8 bosoner i kammer 1 og ett i kammer 2, blir energien i kammer 1 åtte ganger så stor som ovenfor, idet alle de 8 bosonene kan være i den laveste én-partikkel-tilstanden, til tross for at de er identiske. Systemet har altså nå en total energi

$$E(a) = 8E_1 + E_2 = \frac{\hbar^2\pi^2}{2m}\left[\frac{8}{a^2} + \frac{1}{(L_x - a)^2} + 16(1/L_y^2 + 1/L_z^2)\right].$$

Denne er minimal når

$$\frac{d}{da}\left[\frac{8}{a^2} + \frac{1}{(L_x - a)^2}\right] = 2\left[\frac{1}{(L_x - a)^3} - \frac{8}{a^3}\right] = 0,$$

som er oppfylt når $a^3 = 8(L_x - a)^3$, dvs når $a = 2(L_x - a)$, dvs for

$$a = a_2 = 2L_x/3.$$

Oppgave 3

a. ♠Med

$$\chi(0) = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} \equiv \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

har vi at $2a_0^* b_0 = \sin\theta$ og $|a_0|^2 - |b_0|^2 = \cos\theta$. Spinnretningen umiddelbart etter målingen er da

$$\langle\,\boldsymbol{\sigma}\,\rangle_0 = \hat{\mathbf{x}}\,\Re\mathrm{e}(2a_0^* b_0) + \hat{\mathbf{y}}\,\Im\mathrm{m}(2a_0^* b_0) + \hat{\mathbf{z}}\,(|a_0|^2 - |b_0|^2) = \hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta.$$

♠Vi regner ut

$$\begin{aligned}
\mathbf{S}\cdot\langle\,\boldsymbol{\sigma}\,\rangle_0\,\chi(0) &= \tfrac{1}{2}\hbar\boldsymbol{\sigma}\cdot(\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\,\chi(0) = \tfrac{1}{2}\hbar\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} \\
&= \tfrac{1}{2}\hbar\begin{pmatrix} \cos\theta\cos\frac{1}{2}\theta + \sin\theta\sin\frac{1}{2}\theta \\ \sin\theta\cos\frac{1}{2}\theta - \cos\theta\sin\frac{1}{2}\theta \end{pmatrix} = \tfrac{1}{2}\hbar\begin{pmatrix} \cos(\theta - \frac{1}{2}\theta) \\ \sin(\theta - \frac{1}{2}\theta) \end{pmatrix} = \tfrac{1}{2}\hbar\,\chi(0).
\end{aligned}$$

Begynnelsestilstanden $\chi(0)$ er altså en egentilstand til $\mathbf{S}\cdot\langle\,\boldsymbol{\sigma}\,\rangle_0$ med egenverdien $\frac{1}{2}\hbar$.

♠Ifølge målepostulatet skal målingen av $\mathbf{S}\cdot\hat{\mathbf{n}}$ gi en av egenverdiene $\pm\frac{1}{2}\hbar$ og etterlate spinnet i den tilsvarende egentilstanden. Vi kan da konkludere med at måleretningen var $\hat{\mathbf{n}} = \langle\,\boldsymbol{\sigma}\,\rangle_0 = \hat{\mathbf{x}}\,\sin\theta + \hat{\mathbf{z}}\,\cos\theta$, og at måleresultatet var $\mathbf{S}\cdot\hat{\mathbf{n}} = +\frac{1}{2}\hbar$.

b. ♠Spinntilstanden for $t > 0$ må være en lineærkombinasjon av de to stasjonære tilstandene:

$$\chi(t) = c_+\chi_+ e^{-i\omega t/2} + c_-\chi_- e^{i\omega t/2} = \begin{pmatrix} c_+ e^{-i\omega t/2} \\ c_- e^{i\omega t/2} \end{pmatrix}.$$

Her har vi brukt at energiegenverdiene er $E_\pm = \omega S_z = \pm\frac{1}{2}\hbar\omega$. For $t = 0$ har vi da

$$\chi(0) = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix} \qquad \Longrightarrow \qquad c_+ = \cos\tfrac{1}{2}\theta, \quad c_- = \sin\tfrac{1}{2}\theta,$$

slik at tilstanden for $t \geq 0$ er

$$\chi(t) = \begin{pmatrix} \cos\frac{1}{2}\theta\, e^{-i\omega t/2} \\ \sin\frac{1}{2}\theta\, e^{i\omega t/2} \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix}.$$

Med $2a^*b = \sin\theta e^{i\omega t}$ og $|a|^2 - |b|^2 = \cos\theta$ blir spinnretningen ved tiden t da

$$\begin{aligned} \langle\, \boldsymbol{\sigma}\, \rangle_t &= \hat{\mathbf{x}}\, \sin\theta\, \cos\omega t + \hat{\mathbf{y}}\, \sin\theta\, \sin\omega t + \hat{\mathbf{z}}\, \cos\theta \\ &= (\hat{\mathbf{x}}\, \cos\omega t + \hat{\mathbf{y}}\, \sin\omega t)\, \sin\theta + \hat{\mathbf{z}}\, \cos\theta. \end{aligned}$$

Her ser vi at spinnretningen preseserer omkring $\mathbf{B}$-retningen ($\hat{\mathbf{z}}$) med vinkelfrekvensen ω.

c. ♠Fra formelen for $\chi(t)$ ser vi at $\chi(2\pi/\omega) = -\chi(0)$. Dette fortegnsskiftet i spinoren gir akkurat samme $\langle\, \boldsymbol{\sigma}\, \rangle = \chi^\dagger\boldsymbol{\sigma}\chi$ som ved $t = 0$, i overensstemmelse med formelen for $\langle\, \boldsymbol{\sigma}\, \rangle_t$.

♠Analogt med målingen av $\mathbf{S}\cdot\hat{\mathbf{n}} = \mathbf{S}\cdot(\hat{\mathbf{x}}\, \sin\theta + \hat{\mathbf{z}}\cos\theta) = \frac{1}{2}\hbar$ etterlater systemet i tilstanden

$$\chi(0) = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix},$$

vil målingen av $\mathbf{S}\cdot\hat{\mathbf{n}}' = \mathbf{S}\cdot(\hat{\mathbf{x}}\, \sin\theta' + \hat{\mathbf{z}}\cos\theta') = \frac{1}{2}\hbar$ etterlate systemet i tilstanden

$$\chi_{\text{etter}} = \begin{pmatrix} \cos\frac{1}{2}\theta' \\ \sin\frac{1}{2}\theta' \end{pmatrix}.$$

♠Sannsynlighetsamplituden for det siste resultatet er projeksjonen av tilstanden før målingen på tilstanden etter målingen, dvs

$$A = \chi^\dagger_{\text{etter}}\chi_{\text{før}} = (\cos\tfrac{1}{2}\theta' \;\; \sin\tfrac{1}{2}\theta') \begin{pmatrix} -\cos\frac{1}{2}\theta \\ -\sin\frac{1}{2}\theta \end{pmatrix} = -\cos(\tfrac{1}{2}\theta' - \tfrac{1}{2}\theta).$$

Sannsynligheten er altså

$$P = |A|^2 = \cos^2(\tfrac{1}{2}\theta' - \tfrac{1}{2}\theta) = \tfrac{1}{2}[1 + \cos(\theta' - \theta)] = \tfrac{1}{2}[1 + \hat{\mathbf{n}}'\cdot\hat{\mathbf{n}}].$$

Oppgave 4

a. ♠Ved å bruke egenverdiligningen og dennes adjungerte, $\langle\Psi|a_x^\dagger = \alpha^*\langle\Psi|$, finner vi at forventningsverdien av x i tilstanden $|\Psi\rangle$ er

$$\langle\, x\, \rangle = \sqrt{\frac{\hbar}{2m\omega}}\, \langle\Psi|(a_x^\dagger + a_x)|\Psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\, (\alpha^* + \alpha), \quad \text{q.e.d.}$$

Tilsvarende er

$$\langle\, p_x \,\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle\,\Psi\,|(a_x^\dagger - a_x)|\,\Psi\,\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\,(\alpha^* - \alpha), \quad \text{q.e.d.}$$

♠Vha kommutator-relasjonen $a_x a_x^\dagger = 1 + a_x^\dagger a_x$ har vi at

$$\widehat{x}^2 = \frac{\hbar}{2m\omega}(a_x^\dagger + a_x)(a_x^\dagger + a_x) = \frac{\hbar}{2m\omega}(a_x^\dagger a_x^\dagger + a_x a_x + 2a_x^\dagger a_x + 1),$$

og tilsvarende

$$\widehat{p}_x^2 = -\frac{m\hbar\omega}{2}(a_x^\dagger a_x^\dagger + a_x a_x - 2a_x^\dagger a_x - 1).$$

Dette gir

$$\left\langle\, x^2 \,\right\rangle = \frac{\hbar}{2m\omega}[(\alpha^*)^2 + \alpha^2 + 2\alpha^*\alpha + 1] = \frac{\hbar}{2m\omega}[(\alpha^* + \alpha)^2 + 1]$$

og tilsvarende

$$\left\langle\, \widehat{p}_x^2 \,\right\rangle = \frac{m\hbar\omega}{2}[1 - (\alpha^* - \alpha)^2].$$

Vha disse resultatene finner vi usikkerhetene

$$\Delta x = \sqrt{\langle\, x^2 \,\rangle - \langle\, x \,\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \qquad \text{og} \qquad \Delta p_x = \sqrt{\frac{m\hbar\omega}{2}},$$

som gir $\Delta x \Delta p_x = \frac{1}{2}\hbar$.

b. ♠Sannsynlighetstettheten i begynnelsestilstanden,

$$|\Psi_b(x,y,0)|^2 = C_0^4 e^{-m\omega(x-b)^2/\hbar} e^{-m\omega y^2/\hbar},$$

er symmetrisk mhp linjen $x = b$ og mhp linjen $y = 0$. Følgelig er

$$\langle\, x \,\rangle_0 = b \qquad \text{og} \qquad \langle\, y \,\rangle_0 = 0.$$

♠Vi regner ut

$$\begin{aligned} a_x^{\text{pr}}\Psi_b(x,y,0) &= \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\Psi_b(x,y,0) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{\hbar}{m\omega}(m\omega(x-b)/\hbar)\right)\Psi_b(x,y,0) \\ &= \sqrt{\frac{m\omega}{2\hbar}}\, b\,\Psi_b(x,y,0). \end{aligned}$$

Begynnelsestilstanden er altså ganske riktig en egentilstand til a_x^{pr} med egenverdien

$$\alpha_x(0) = \sqrt{\frac{m\omega}{2\hbar}}\, b.$$

Tilsvarende er

$$\begin{aligned}
a_y^{\mathrm{pr}}\Psi_b(x,y,0) &= \sqrt{\frac{m\omega}{2\hbar}}\left(y+\frac{\hbar}{m\omega}\frac{\partial}{\partial y}\right)\Psi_b(x,y,0) \\
&= \sqrt{\frac{m\omega}{2\hbar}}\left(y+\frac{\hbar}{m\omega}(-m\omega y/\hbar + im\omega b/\hbar)\right)\Psi_b(x,y,0) \\
&= i\sqrt{\frac{m\omega}{2\hbar}}\,b\,\Psi_b(x,y,0).
\end{aligned}$$

Egenverdien til a_y^{pr} er altså

$$\alpha_y(0) = i\sqrt{\frac{m\omega}{2\hbar}}\,b = i\alpha_x(0), \quad \text{q.e.d.}$$

c. ♠Fra resultatene ovenfor har vi

$$\langle\, x\,\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha_x(t)+\alpha_x^*(t)\right] = \sqrt{\frac{\hbar}{2m\omega}}\cdot 2\Re\mathrm{e}\left(\alpha_x(0)\,e^{-i\omega t}\right) = b\cos\omega t$$

og

$$\langle\, y\,\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha_y(t)+\alpha_y^*(t)\right] = \sqrt{\frac{\hbar}{2m\omega}}\cdot 2\Re\mathrm{e}\left(i\alpha_x(0)\,e^{-i\omega t}\right) = b\sin\omega t.$$

Som en kontroll ser vi at verdiene for $t=0$ stemmer med resultatene i pkt. **b**.

♠Da

$$\langle\, x\,\rangle_t^2 + \langle\, y\,\rangle_t^2 = b^2,$$

ser vi at forventningsverdien av posisjonen beveger seg på en sirkelbane med radius b, og med vinkelhastighet ω.

♠Vi noterer først at produktløsningen reproduserer den korrekte begynnelsestilstanden. Da gjenstår det bare å vise at den oppfyller Schrödingerligningen: Innsetting gir

$$\left[i\hbar\frac{\partial}{\partial t}-\widehat{H}\right]\Psi_x(x,t)\Psi_y(y,t) = \left[i\hbar\frac{\partial}{\partial t}-\widehat{H}^{(x)}-\widehat{H}^{(y)}\right]\Psi_x(x,t)\Psi_y(y,t)$$

$$\begin{aligned}
&= \left[i\hbar\frac{\partial}{\partial t}\,\Psi_x(x,t)\right]\Psi_y(y,t) + \Psi_x(x,t)i\hbar\frac{\partial}{\partial t}\,\Psi_y(y,t) \\
&\quad - \left[\widehat{H}^{(x)}\Psi_x(x,t)\right]\Psi_y(y,t) - \Psi_x(x,t)\,\widehat{H}^{(y)}\Psi_y(y,t) \\
&= \left[\left(i\hbar\frac{\partial}{\partial t}-\widehat{H}^{(x)}\right)\Psi_x(x,t)\right]\Psi_y(y,t) + \Psi_x(x,t)\left(i\hbar\frac{\partial}{\partial t}-\widehat{H}^{(y)}\right)\Psi_y(y,t) \\
&= 0, \quad \text{q.e.d.}
\end{aligned}$$