TFY 4250 / FY 2045 – QUM 1

Final Exam 2013

– Solutions –

Note: Each of the 6 problems is worth 10 marks.

$\Rightarrow$ Total: 60 marks

QUM I – Final Exam 2013 (Each problem is worth 10 marks.)

Solutions

1) We compute

$$\frac{d}{dt}\langle F\rangle = \frac{d}{dt}\int \psi^* \hat{F}\psi \, dx$$

$$= \int\left[\left(\frac{\partial \psi^*}{\partial t}\right)\hat{F}\psi + \psi^* \frac{\partial \hat{F}}{\partial t}\psi + \psi^* \hat{F}\frac{\partial \psi}{\partial t}\right] dx$$

Using SE: $i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi$

$$= \frac{i}{\hbar}\int (\hat{H}\psi)^* \hat{F}\psi \, dx + \int \psi^* \frac{\partial \hat{F}}{\partial t}\psi \, dx - \frac{i}{\hbar}\int \psi^* \hat{F}\hat{H}\psi \, dx$$

$\hat{H}$ Hermitian

$$= \frac{i}{\hbar}\int \psi^* \hat{H}\hat{F}\psi \, dx + \int \psi^* \frac{\partial \hat{F}}{\partial t}\psi \, dx - \frac{i}{\hbar}\int \psi^* \hat{F}\hat{H}\psi \, dx$$

$$= \frac{i}{\hbar}\int \psi^* [\hat{H},\hat{F}]\psi \, dx + \int \psi^* \frac{\partial \hat{F}}{\partial t}\psi \, dx$$

$$= \frac{i}{\hbar}\langle [\hat{H},\hat{F}]\rangle + \left\langle \frac{\partial \hat{F}}{\partial t}\right\rangle \quad \square$$

2) a) We compute $\langle H \rangle$ according to

$$\langle H \rangle = \frac{\int_0^\infty \varphi^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi(x)\,dx}{\int_0^\infty \varphi^*(x)\,\varphi(x)\,dx}$$

Let us start with the denominator:

$$\int_0^\infty \varphi^*(x)\,\varphi(x)\,dx = |A|^2 \int_0^\infty x^2 e^{-2ax}\,dx$$

$$= |A|^2\Big(\underbrace{-\frac{1}{2a}e^{-2ax}x^2\Big|_0^\infty}_{=0} + \int_0^\infty \frac{1}{2a}e^{-2ax}\,2x\,dx\Big)$$

$$= |A|^2\,\frac{2}{2a}\Big(\underbrace{-\frac{1}{2a}e^{-2ax}x\Big|_0^\infty}_{=0} + \int_0^\infty \frac{1}{2a}e^{-2ax}\,dx\Big)$$

$$= \frac{|A|^2}{a}\left(-\frac{1}{4a^2}\right)e^{-2ax}\Big|_0^\infty$$

$$= \frac{|A|^2}{4a^3}.$$

The numerator turns into

$$|A|^2\int_0^\infty xe^{-ax}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\gamma x\right)xe^{-ax}\,dx$$

$$\left(\frac{d}{dx}\left(xe^{-ax}\right) = (1-ax)e^{-ax} \Rightarrow \frac{d^2}{dx^2}\left(xe^{-ax}\right) = (-2a+a^2x)e^{-ax}\right)$$

$$= |A|^2\int_0^\infty xe^{-ax}\left(\frac{\hbar^2}{2m}(2a-a^2x)e^{-ax}+\gamma x^2e^{-ax}\right)dx$$

$$= |A|^2\int_0^\infty e^{-2ax}\left(\frac{\hbar^2}{2m}(2ax-a^2x^2)+\gamma x^3\right)dx$$

$$= |A|^2\Big[\underbrace{\int_0^\infty e^{-2ax}\left(\frac{\hbar^2}{2m}2ax\right)}_{1} - \underbrace{\int_0^\infty e^{-2ax}\left(+\frac{\hbar^2}{2m}a^2x^2\right)dx}_{2} + \underbrace{\int_0^\infty e^{-2ax}\gamma x^3\,dx}_{3}\Big]$$

Integral ①: $\displaystyle\int_0^\infty \frac{\hbar^2}{m} a x e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \frac{\hbar^2}{m} a x \Big|_0^\infty}_{=0} + \int_0^\infty \frac{\hbar^2}{2m} e^{-2ax} dx$

$\displaystyle = -\frac{1}{2a}\left(\frac{\hbar^2}{2m}\right) e^{-2ax} \Big|_0^\infty$

$\displaystyle = \frac{\hbar^2}{4am}$

Integral ②: $\displaystyle\int_0^\infty \frac{\hbar^2}{2m} a^2 x^2 e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \frac{\hbar^2}{2m} a^2 x^2 \Big|_0^\infty}_{=0} + \int_0^\infty \frac{\hbar^2}{2m} a x e^{-2ax} dx$

$\displaystyle = \underbrace{-\frac{1}{2a} e^{-2ax} \frac{\hbar^2}{2m} a x \Big|_0^\infty}_{=0} + \int_0^\infty \frac{\hbar^2}{4m} e^{-2ax} dx$

$\displaystyle = -\frac{\hbar^2}{8am} e^{-2ax} \Big|_0^\infty = \frac{\hbar^2}{8am}$

Integral ③: $\displaystyle\int_0^\infty \gamma x^3 e^{-2ax} dx = \underbrace{-\frac{1}{2a} e^{-2ax} \gamma x^3 \Big|_0^\infty}_{=0} + \int_0^\infty \frac{3\gamma}{2a} x^2 e^{-2ax} dx$

$\displaystyle = \underbrace{-\frac{3}{4a^2} \gamma x^2 e^{-2ax} \Big|_0^\infty}_{=0} + \int_0^\infty \frac{3\gamma}{2a^2} x e^{-2ax} dx$

$\displaystyle = \underbrace{-\frac{3\gamma}{4a^3} x e^{-2ax} \Big|_0^\infty}_{=0} + \int_0^\infty \frac{3\gamma}{4a^3} e^{-2ax} dx$

$\displaystyle = -\frac{3\gamma}{8a^4} e^{-2ax} \Big|_0^\infty = \frac{3\gamma}{8a^4}$

Hence, we obtain

$$\langle H \rangle = \frac{\frac{\hbar^2}{4am} - \frac{\hbar^2}{8am} + \frac{3\gamma}{8a^4}}{1/4a^3} \qquad (|A|^2 \text{ cancels out})$$

$$\Rightarrow \boxed{\langle H \rangle = \frac{\hbar^2}{2m} a^2 + \frac{3\gamma}{2a}} \qquad (*)$$

b) We need $\varphi(0) = 0$ and $\varphi(x) \xrightarrow[x \to \infty]{} 0$.

$\varphi(x)$ fulfills these criteria.

c) Now, we need to minimize $\langle H \rangle$ with respect to a.

$$\Rightarrow \quad \frac{d\langle H \rangle}{da} \stackrel{!}{=} 0$$

$$\Rightarrow \quad \frac{\hbar^2}{m} a - \frac{3\gamma}{2a^2} = 0$$

$$\Rightarrow \quad \frac{\hbar^2}{m} a^3 = \frac{3\gamma}{2}$$

$$\Rightarrow \quad a = \left(\frac{3\gamma m}{2\hbar^2}\right)^{1/3}$$

Substitution into $(*)$ gives

$$\langle H \rangle_\varphi^{min} = \frac{\hbar^2}{2m}\left(\frac{3\gamma m}{2\hbar^2}\right)^{2/3} + \frac{3}{2}\gamma\left(\frac{2\hbar^2}{3\gamma m}\right)^{1/3}$$

$$= \frac{3}{4}\left(\frac{2\gamma^2\hbar^2}{3m}\right)^{1/3} + \frac{3}{2}\left(\frac{2\gamma^2\hbar^2}{3m}\right)^{1/3}$$

$$\Rightarrow \boxed{\langle H \rangle_\varphi^{min} = \frac{9}{4}\left(\frac{2\gamma^2\hbar^2}{3m}\right)^{1/3}} \geq E_0$$

3) a) The energy corrections at first order are:

$$E_n^{(1)} = \langle n | H_1 | n \rangle = \langle \psi_n^{(0)} | H_1 | \psi_n^{(0)} \rangle$$

$$= -F \langle \psi_n^{(0)} | x | \psi_n^{(0)} \rangle = 0$$, according to the formula stated in the problem.

$$\Rightarrow \boxed{E_n^{(1)} = 0}$$

Likewise, the second-order corrections are

$$E_n^{(2)} = \sum_m^{m \neq n} \frac{|\langle \psi_m^{(0)} | H_1 | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}} = F^2 \sum_m^{m \neq n} \frac{|\langle \psi_m^{(0)} | x | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

But we know that

$$|\langle \psi_m^{(0)} | x | \psi_n^{(0)} \rangle|^2 = \frac{\hbar}{2m\omega} \left[(n+1)\, \delta_{m,n+1} + n\, \delta_{m,n-1} \right].$$

Therefore, we obtain

$$E_n^{(2)} = \frac{F^2 \hbar}{2m\omega} \left(\frac{n+1}{-\hbar\omega} + \frac{n}{\hbar\omega} \right) = \frac{F^2}{2m\omega^2} \left(n - (n+1) \right)$$

$$\Rightarrow \boxed{E_n^{(2)} = -\frac{F^2}{2m\omega^2}}$$

b) We write the Hamiltonian as

$$H = H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - Fx$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x^2 - \frac{2Fx}{m\omega^2 x} \right)$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x^2 - 2x \frac{F}{m\omega^2} + \frac{F^2}{m^2\omega^4} \right) - \frac{F^2}{2m\omega^2}$$

$$= \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x - \frac{F}{m\omega^2} \right)^2 - \frac{F}{2m\omega^2}$$

Introducing the new variables $p = \bar{p}$, $\bar{x} = x - \frac{F}{m\omega^2}$,

we obtain:

$$\bar{H} = \frac{\bar{p}^2}{2m} + \frac{1}{2} m\omega^2 \bar{x}^2 - \frac{F^2}{2m\omega^2}.$$

This problem has the (exact) "eigen energies":

$$E_n^{exact} = \left(n + \tfrac{1}{2}\right)\hbar\omega - \frac{F^2}{2m\omega^2} = \underline{\underline{E_n^{(0)} + E_n^{(2)}}}$$

We see that the exact correction equals the second-order correction.

4) a) We require

$$1 \stackrel{!}{=} \chi^{*}\chi = |A|^2 (1+4+4) = 9\,|A|^2 \Rightarrow A = 1/3$$

(we choose A to be real and positive)

b) Since we are dealing with a spin-1/2 system, a measurement would obviously yield the value $\pm \hbar/2$.

Let us denote the spin-up state by $\chi_z^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the spin-down state by $\chi_z^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By definition, ~~the~~ probability of finding the ~~state~~ system in the state χ_z^{+} is:

$$|\chi_z^{+\dagger} \chi|^2 = \left| \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dagger} \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{5}{9} \text{ ; measured value} = \hbar/2$$

Likewise, we obtain the probability of finding the system in the spin-down state:

$$|\chi_z^{-\dagger} \chi|^2 = \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\dagger} \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{4}{9} \text{ ; measured value} = -\hbar/2$$

Also, $\langle S_z \rangle = \langle \chi | S_z | \chi \rangle = \hbar/18$.

c) Using $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we find the spin-up and spin-down states w.r.t. the x-axis as:

$$\chi_x^{+} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad \chi_x^{-} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The measurements would again yield values $\pm \hbar/2$.
The measurements and probabilities now read:

$$+\hbar/2 : \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\dagger} \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{13}{18}$$

$$-\hbar/2 : \left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\dagger} \cdot \frac{1}{3} \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2 = \frac{5}{18}$$

Also: $\langle S_x \rangle = \langle \chi | S_x | \chi \rangle = \frac{2}{9}\hbar$

5) a) The energy E is determined by the kinetic energy alone:

$$\hat{H}\psi_{n_x n_y n_z} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) A\cdot \sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_z\pi z}{L}\right)$$

$$= \frac{\hbar^2}{2m} A\cdot\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2+n_z^2}{L^2}\right)\pi^2 \sin\left(\frac{n_x\pi x}{L_x}\right)\sin\left(\frac{n_y\pi y}{L}\right)\sin\left(\frac{n_z\pi z}{L}\right)$$

$$= \underbrace{\frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2+n_z^2}{L^2}\right)}_{=E}\cdot\psi_{n_x n_y n_z}$$

$$\Rightarrow \boxed{E = \frac{\hbar^2\pi^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2+n_z^2}{L^2}\right)}, \quad n_x, n_y, n_z \in \mathbb{N}.$$

We use the relation for the change in energy:

$$F_x \cdot dL_x = -dE$$

$$\Rightarrow \quad F_x = -\frac{\partial E}{\partial L_x} = \frac{\hbar^2\pi^2 n_x^2}{mL_x^3} \underset{\uparrow}{=} \frac{\hbar^2\pi^2}{mL_x^3} \quad \square$$

for $n_x=1$ (ground state)

b) Eight identical spin-1/2 fermions can be described by the following quantum numbers :

$$(n_x, n_y, n_z) = (1,1,1), (2,1,1), (1,2,1), (1,1,2).$$

Here, we assume that the system is in its ground state. Note that each state (n_x, n_y, n_z) describes a one-particle state in which two fermions with opposite spin are found, for a total of $2 \times 4 = 8$ states.

The force depends only on the L_x-dependent contributions to the total energy:

$$E_{tot}^{(x)} = \frac{\hbar^2 \pi^2}{2m} \cdot \underset{\substack{\uparrow \\ \text{spin up/down}}}{2} \cdot \Big(\overbrace{\frac{1}{L_x^2} + \frac{4}{L_x^2} + \frac{1}{L_x^2} + \frac{1}{L_x^2}}^{n_x} \Big) = 7 \frac{\hbar^2 \pi^2}{m L_x^2}.$$

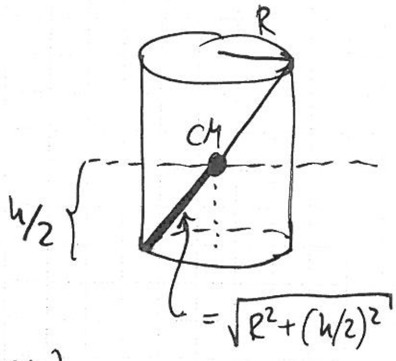
The force is then

$$\boxed{F_x = - \left.\frac{\partial E_{tot}^{(x)}}{\partial L_x}\right|_L = - \left.\frac{\partial E_{tot}}{\partial L_x}\right|_L = 14 \frac{\hbar^2 \pi^2}{m L^3}}.$$

6) First, we need to compute

$$\gamma = \frac{1}{\hbar}\int_0^{x_0} |p(x)|\,dx = \frac{1}{\hbar}\int_0^{x_0}\sqrt{2m(V-E)}$$

$$\underset{\underline{E=0}}{=} \frac{1}{\hbar}\int_0^{x_0}\sqrt{2m\,mgx} \qquad \underline{V=mgx}$$



The can topples when the centre of mass (CM) is above the point of rotation, i.e. at maximum potential energy. At that moment, we have

$$x_0 = \sqrt{R^2+(h/2)^2} - h/2 \;, \qquad (x_0 = 0.83\text{ cm})$$

where x is the height of CM relative to its equilibrium value.

$$\Rightarrow \gamma = \frac{\sqrt{2m}}{\hbar}\sqrt{mg}\int_0^{x_0}\sqrt{x}\,dx = \frac{m}{\hbar}\sqrt{2g}\,\frac{2}{3}\,x^{3/2}\Big|_0^{x_0} = \frac{2m}{3\hbar}\sqrt{2g}\,x_0^{3/2}$$

$$\Rightarrow \gamma \approx \frac{2\cdot 0.3\,\text{Kg}}{3\cdot 1.05\cdot 10^{-34}\,\text{Js}}\sqrt{2\cdot 9.8\,\text{m/s}^2}\cdot(0.0083\text{ m})^{3/2} = 6.4\cdot 10^{30}$$

The lifetime is approximated by

$$\tau = \frac{2R}{v}\,e^{2\gamma}.$$

The velocity equals the thermal velocity:

$$\frac{1}{2}mv^2 = \frac{1}{2}k_BT \Rightarrow v = \sqrt{\frac{k_BT}{m}}$$

Then, we finally obtain

$$\tau = \frac{2 \cdot R}{v} e^{2\gamma} = 2R \cdot \sqrt{\frac{m}{k_B T}} \; e^{2\gamma}$$

$$= 2 \cdot (0.03) \sqrt{\frac{0.3}{1.4 \cdot 10^{-23} \cdot 300}} \; e^{12.8 \cdot 10^{30}} \; s$$

$$= 5 \cdot 10^{8} \cdot e^{12.8 \cdot 10^{30}} \; s$$

$$= 16 \cdot e^{12.8 \cdot 10^{30}} \; \text{yrs.}$$

This time vastly exceeds the age of the universe.