NTNU Trondheim, Institutt for fysikk

Examination for FY3452 Gravitation and Cosmology
Contact: Kåre Olaussen, tel. 735 93652 / 45437170
Possible languages for your answers: Bokmål, English, German, Nynorsk.
Allowed tools: Pocket calculator, mathematical tables
Some formulas can be found at the end of p.2.

1. Sphere S^2.

The line-element of the two-dimensional unit sphere S^2 is given by

$$ds^2 = d\vartheta^2 + \sin^2\vartheta d\phi^2 .$$

a. Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a{}_{bc}$. (6 pts)
b. Calculate the Ricci tensor R_{ab} and the scalar curvature R. (Hint: Use the symmetry properties of this space.) (6 pts)

a. We use as Lagrange function L the kinetic energy T. From $L = g_{ab}\dot{x}^a\dot{x}^b = \dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2$ we find

$$\frac{\partial L}{\partial \phi} = 0 \quad , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt}(2\sin^2\vartheta\dot{\phi}) = 2\sin^2\vartheta\ddot{\phi} + 4\cos\vartheta\sin\vartheta\dot{\vartheta}\dot{\phi}$$

$$\frac{\partial L}{\partial \vartheta} = 2\cos\vartheta\sin\vartheta\dot{\phi}^2 \quad , \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = \frac{d}{dt}(2\dot{\vartheta}) = 2\ddot{\vartheta}$$

and thus the Lagrange equations are

$$\ddot{\phi} + 2\cot\vartheta\dot{\vartheta}\dot{\phi} = 0 \qquad \text{and} \qquad \ddot{\vartheta} - \cos\vartheta\sin\vartheta\dot{\phi}^2 = 0 .$$

Comparing with the given geodesic equation, we read off the non-vanishing Christoffel symbols as $\Gamma^\phi{}_{\vartheta\phi} = \Gamma^\phi{}_{\phi\vartheta} = \cot\vartheta$ and $\Gamma^\vartheta{}_{\phi\phi} = -\cos\vartheta\sin\vartheta$. (Remember that $2\cot\vartheta = \Gamma^\phi{}_{\vartheta\phi} + \Gamma^\phi{}_{\phi\vartheta}$.)

b. The Ricci tensor of a maximally symmetric spaces satisfies $R_{ab} = Kg_{ab}$. Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too, $R_{\phi\vartheta} = R_{\vartheta\phi} = 0$. We calculate with

$$R_{ab} = R^c{}_{acb} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{ab}\Gamma^d{}_{cd} - \Gamma^d{}_{bc}\Gamma^c{}_{ad}$$

the $\vartheta\vartheta$ component,

$$\begin{aligned} R_{\vartheta\vartheta} &= 0 - \partial_\vartheta(\Gamma^\phi{}_{\vartheta\phi} + \Gamma^\vartheta{}_{\vartheta\vartheta}) + 0 - \Gamma^d{}_{\vartheta c}\Gamma^c{}_{\vartheta d} = 0 + \partial_\vartheta\cot\vartheta - \Gamma^\phi{}_{\vartheta\phi}\Gamma^\phi{}_{\vartheta\phi} \\ &= 0 - \partial_\vartheta\cot\vartheta - \cot^2\vartheta = 1 \end{aligned}$$

From $R_{ab} = Kg_{ab}$, we find $R_{\vartheta\vartheta} = Kg_{\vartheta\vartheta}$ and thus $K = 1$. Hence $R_{\phi\phi} = g_{\phi\phi} = \sin^2\vartheta$.
The scalar curvature is (diagonal metric with $g^{\phi\phi} = 1/\sin^2\vartheta$ and $g^{\vartheta\vartheta} = 1$)

$$R = g^{ab}R_{ab} = g^{\phi\phi}R_{\phi\phi} + g^{\vartheta\vartheta}R_{\vartheta\vartheta} = \frac{1}{\sin^2\vartheta}\sin^2\vartheta + 1 \times 1 = 2 .$$

[If you wonder that $R = 2$, not 1: in $d = 2$, the Gaussian curvature K is connected to the "general" scalar curvature R via $K = R/2$. Thus $K = \pm 1$ means $R = \pm 2$ for spaces of constant unit curvature radius, S^2 and H^2.]

2. Black holes.
The metric outside a spherically symmetric mass distribution with mass M is given in Schwarzschild coordinates by

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-\frac{2M}{r}} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2) - \mathrm{d}t^2\left(1-\frac{2M}{r}\right)$$

a. Use the "advanced time parameter"

$$p = t + r + 2M\ln|r/2M - 1|$$

to eliminate t in the line-element (i.e. introduce Eddington-Finkelstein coordinates) and show that in the new coordinates the singularity at $R = 2M$ is absent. (3 pts)
b. Draw a space-time diagram considering radial light-rays in the $\tilde{t} \equiv p - r, r$ plane. Include the world-line of an observer falling into the black hole. Explain why $r = 2M$ is an event horizon. (4 pts)
c. Determine the smallest possible stable circular orbit of a massive particle. (Hint: Use the Killing vectors of the metric and consider the effective potential V_{eff}.) (7 pts)

a. Forming the differential,

$$\mathrm{d}p = \mathrm{d}t + \mathrm{d}r + \left(\frac{r}{2M} - 1\right)^{-1}\mathrm{d}r = \mathrm{d}t + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r\,,$$

we can eliminate $\mathrm{d}t$ from the Schwarzschild metric and find

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}r + r^2\mathrm{d}\Omega\,.$$

This metric is regular at $2M$ and valid for all $r > 0$.

b. For radial light-rays, $\mathrm{d}s = \mathrm{d}\phi = \mathrm{d}\vartheta = 0$, it follows

$$0 = -\left(1-\frac{2M}{r}\right)\mathrm{d}p^2 + 2\mathrm{d}p\mathrm{d}r\,.$$

There exist three types solutions: i) for $r = 2M$, light-rays have constant r and p; ii) light-rays with $p =$ const.; iii) dividing by $\mathrm{d}p$,

$$0 = -\left(1-\frac{2M}{r}\right)\mathrm{d}p + \mathrm{d}r$$

we separate variables and integrate,

$$p - 2(r + 2M\ln|r/2M - 1|) = \text{const.}$$

The light-rays of type ii) are ingoing: as t increase, r has to increase to keep p constant. The light-rays of type ii) are ingoing for $r < 2M$ and outgoing for $r > 2M$. Thus for $r < 2M$ both radial light-rays moves towards $r = 0$; all wordlines of observers are inside such light-cones and have to move towards $r = 0$ too. Hence $r = 2M$ is an event horizon.

c. Spherical symmetry allows us to choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. Then we replace in the normalization condition $\mathbf{u} \cdot \mathbf{u} = -1$ written out for the Schwarzschild metric,

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2$$

the velocities u_t and u_r by the conserved quantities

$$\begin{aligned} e &\equiv -\xi \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ l &\equiv \eta \cdot \mathbf{u} = r^2 \sin\vartheta^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \end{aligned}$$

Inserting e and l, then reordering gives

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}$$

with

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}.$$

Circular orbits correspond to $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ with

$$r_{1,2} = \frac{l^2}{2M}\left[1 \pm \sqrt{1 - 12M^2/l^2}\right].$$

The stable circular orbit (i.e. at the minimum of V_{eff}) corresponds to the plus sign. The square root becomes negative for $l^2 = 6M$ and thus the "innermost stable circular orbit" is for a Schwarzschild black hole at $r_{\text{ISCO}} = 6M$.

3. Cosmology.

Consider a flat universe dominated by one matter component with E.o.S. $w = P/\rho$ =const.
a. Use that the universe expands adiabatically to find the connection $\rho = \rho(R, w)$ between the density ρ, the scale factor R and the state parameter w. (4 pts)
b. Find the age t_0 of the universe as function of w and the current value of the Hubble parameter, H_0. (3 pts)
c. Comment on the value of t_0 in the case of a positive cosmological constant, $w = -1$. (2 pts)
d. Find the relative energy loss per time, $E^{-1}\,\mathrm{d}E/\mathrm{d}t$, of relativistic particles due to the expansion of the universe for $H_0 = 70$km/s/Mpc. (1 pt)

a. For adiabatic expansion, the first law of thermodynmaics becomes $\mathrm{d}U = -P\mathrm{d}V$ or

$$\mathrm{d}(\rho R^3) = -3PR^2\mathrm{d}R$$

Eliminating P with $P = P(\rho) = w\rho$,

$$\frac{\mathrm{d}\rho}{\mathrm{d}R} R^3 + 3\rho R^2 = -3w\rho R^2 \,.$$

Separating the variables,

$$-3(1+w)\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\rho}{\rho} \,,$$

we can integrate and obtain $\rho \propto R^{-3(1+w)}$.

b. For a flat universe, $k = 0$, with one dominating energy component with $w = P/\rho = \text{const.}$ and $\rho = \rho_{\text{cr}}\,(R/R_0)^{-3(1+w)}$, the Friedmann equation becomes

$$\dot{R}^2 = \frac{8\pi}{3} G\rho R^2 = H_0^2 R_0^{3+3w}\; R^{-(1+3w)} \,, \tag{1}$$

where we inserted the definition of $\rho_{\text{cr}} = 3H_0^2/(8\pi G)$. Separating variables we obtain

$$R_0^{-(3+3w)/2} \int_0^{R_0} \mathrm{d}R\, R^{(1+3w)/2} = H_0 \int_0^{t_0} \mathrm{d}t = t_0 H_0 \tag{2}$$

and hence the age of the Universe follows as

$$t_0 H_0 = \frac{2}{3+3w} \,.$$

c. Models with $w > -1$ need a finite time to expand from the initial singularity $R(t=0) = 0$ to the current value of the scale factor R_0, while a Universe with only a Λ has no "beginning", $t_0 H_0 \to \infty$.

d. The connection between the energy E_0 today and the energy at redshift z is

$$E(z) = (1+z)E_0$$

and thus $\mathrm{d}E = \mathrm{d}z E_0$. Differentiating $1 + z = R_0/R(t)$, we obtain with $H = \dot{R}/R$

$$\mathrm{d}z = -\frac{R_0}{R^2}\,\mathrm{d}R = -\frac{R_0}{R^2}\,\frac{\mathrm{d}R}{\mathrm{d}t}\,\mathrm{d}t = -(1+z)H\mathrm{d}t \,.$$

Combining the two equations, we find $\mathrm{d}E = -(1+z)H\mathrm{d}tE_0 = -H\mathrm{d}tE$ or

$$\frac{1}{E}\,\frac{\mathrm{d}E}{\mathrm{d}t} = -H(z) = -H_0(1+z)^{3/2} \,.$$

Numerically, we find for the current epoch

$$\frac{1}{E}\,\frac{dE}{dt} \approx \frac{7.1\times 10^6\text{cm}}{\text{s}\;3.1\times 10^{24}\text{cm}} \approx 5.2\times 10^{-36}\text{s}^{-1} \,.$$

4. Symmetries.
Consider in Minkowski space a complex scalar field ϕ with Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_a\phi^\dagger\partial^a\phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2 .$$

a. Name the symmetries of the Langrangian. (1.5 pts)
b. Derive Noether's theorem in the form

$$0 = \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a - K^\mu\right) .$$

(4.5 pts)
c. Derive one conserved current of your choice. (4 pts)

a. space-time symmetries: Translation, Lorentz, scale invariance. internal: global SO(2) / U(1) invariance.

b. We assume that the collection of fields ϕ_a has a continuous symmetry group. Thus we can consider an infinitesimal change $\delta\phi_a$ that keeps $\mathcal{L}(\phi_a, \partial_\mu\phi_a)$ invariant,

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\partial_\mu\phi_a . \tag{3}$$

Now we exchange $\delta\partial_\mu$ against $\partial_\mu\delta$ in the second term and use then the Lagrange equations, $\delta\mathcal{L}/\delta\phi_a = \partial_\mu(\delta\mathcal{L}/\delta\partial_\mu\phi_a)$, in the first term. Then we can combine the two terms using the Leibniz rule,

$$0 = \delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial_\mu\delta\phi_a = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a\right) . \tag{4}$$

Hence the invariance of $\mathcal{L}$ under the change $\delta\phi_a$ implies the existence of a conserved current, $\partial_\mu J^\mu = 0$, with

$$J_1^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a . \tag{5}$$

If the transformation $\delta\phi_a$ leads to change in $\mathcal{L}$ that is a total four-divergence, $\delta\mathcal{L} = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equation of motions are still invariant. The conserved current is changed to

$$J^\mu = \delta\mathcal{L}/\delta\partial_\mu\phi_a\,\delta\phi_a - K^\mu .$$

c. i) Translations: From $\phi_a(x) \to \phi_a(x-\varepsilon) \approx \phi_a(x) - \varepsilon^\mu\partial_\mu\phi(x)$ we find the change $\delta\phi_a(x) = -\varepsilon^\mu\partial_\mu\phi(x)$. The Lagrange density changes similiarly, $\mathcal{L}(x) \to \mathcal{L}(x-\varepsilon)$ or $\delta\mathcal{L}(x) = -\varepsilon^\mu\partial_\mu\mathcal{L}(x) = -\partial_\mu(\varepsilon^\mu\mathcal{L}(x))$. Thus $K^\mu = -\varepsilon^\mu\mathcal{L}(x)$ and inserting both in the Noether current gives

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,[-\varepsilon^\nu\partial_\nu\phi(x)] + \varepsilon^\mu\mathcal{L}(x) = \varepsilon_\nu T^{\mu\nu}$$

with $T^{\mu\nu}$ as energy-momentum tensor and four-momentum as Noether charge.
or
ii) Charge conservation: We can work either with complex fields and U(1) phase transformations

$$\phi(x) \to \phi(x)\mathrm{e}^{\mathrm{i}\alpha} \quad , \quad \phi^\dagger(x) \to \phi^\dagger(x)\mathrm{e}^{-\mathrm{i}\alpha}$$

or real fields (via $\phi = (\phi + \mathrm{i}\phi_2)/\sqrt{2}$) and invariance under rotations SO(2). With $\delta\phi = \mathrm{i}\alpha\phi$, $\delta\phi^\dagger = -\mathrm{i}\alpha\phi^\dagger$, the conserved current is

$$J^\mu = \mathrm{i}\left[\phi^\dagger\partial^\mu\phi - (\partial^\mu\phi^\dagger)\phi\right]$$

Some formula: Signature of the metric $(-,+,+,+)$.

$$\ddot{x}^c + \Gamma^c{}_{ab}\dot{x}^a\dot{x}^b = 0$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}\,,$$

$$\frac{e^2-1}{2} = \frac{\dot{r}^2}{2} + V_{\text{eff}}$$

$$H^2 = \frac{8\pi}{3}G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)$$

$$1\text{Mpc} = 3.1 \times 10^{24}\text{cm}$$

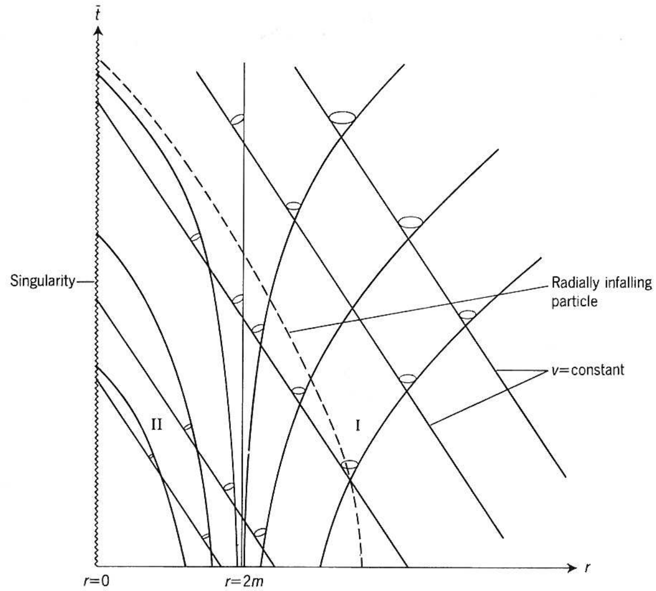

Fig. 16.10 Schwarzschild solution in advanced Eddington–Finkelstein coordinates.