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Solution to the exam in FY3452 GRAVITATION OG COSMOLOGY

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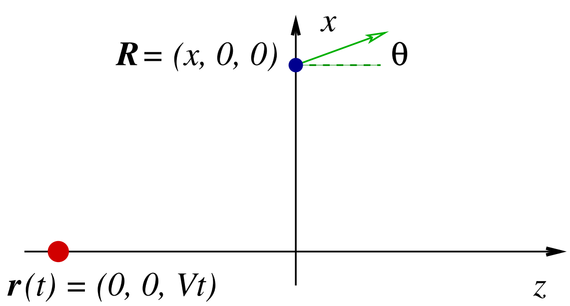
This solution consists of 7 pages.

Problem 1. Lorentz transformations



An object is moving with constant velocity V along the z-axis, i.e. along the curve

$$\boldsymbol{r}(t) = Vt\,\hat{\boldsymbol{e}}_z. \tag{1}$$

It is observed by detectors at rest at the position $\boldsymbol{R} = x\,\hat{\boldsymbol{e}}_x$. You may assume x to be non-negative. You may also choose to use units where the speed of light $c = 1$.

First assume the object is emitting light (photons) in all directions. Seen from a coordinate system where the object is a rest this light is monochromatic, i.e. all photons have the same energy $\hbar\omega_0$.

a) A photon emitted from the object is observed at position $\boldsymbol{R}$ at time t. At which time t_0 was it emitted? Verify your solution by checking the special cases (i) $x = 0$, and (ii) $t_0 = 0$.

At time t_0 the object was at position $Vt_0\,\hat{\boldsymbol{e}}_z$, at a distance $\ell = \sqrt{V^2t_0^2 + x^2}$ from $\boldsymbol{R}$. I.e., the photon needs a time $t - t_0 = \ell/c$ to travel from emission to detection,

$$c^2(t - t_0)^2 = V^2t_0^2 + x^2. \tag{2}$$

Solving for t_0 gives

$$t_0 = \frac{1}{1-(V/c)^2}\left\{t - \frac{1}{c}\sqrt{(Vt)^2 + [1-(V/c)^2]\,x^2}\right\}. \tag{3}$$

(i) For $x = 0$ the photon needs the time $t - t_0$ to travel a distance Vt_0. It follows that we must have $t_0 = t/[1 + (V/c)]$, which agrees with (3) since $[1-(V/c)]\,/\,\left[1-(V/c)^2\right] = 1/[1+(V/c)]$.

(ii) For $t_0 = 0$ the photon was emitted from the origin ($z = 0$). I.e., the detection must occur at time $t = x/c$. Inserting this value for t into (3) indeed gives $t_0 = 0$.

b) From which position on the z-axis was the photon emitted? In which direction $\hat{\boldsymbol{n}} = \cos\theta\,\hat{\boldsymbol{e}}_z + \sin\theta\,\hat{\boldsymbol{e}}_x$ is the photon observed to move? Express this by finding the quantity $\cot\theta$, with θ as given in the figure above.

The emission occured from position $z = Vt_0$, with t_0 given by (3). The photon is travelling from $(0,0,z)$ to $(x,0,0)$, i.e. at an angle θ such that

$$\cot\theta = -\frac{z}{x} = -\frac{Vt_0}{x} = \frac{(V/x)}{1-(V/c)^2}\left\{\frac{1}{c}\sqrt{(Vt)^2 + [\,1-(V/c)^2\,]\,x^2} - t\right\}. \tag{4}$$

c) What is the egenvelocity u'^{ν} of the object in a coordinate system where it is at rest? What is the eigenvelocity u^{μ} of the object in our coordinate system (where the detectors are at rest)?

The egenvelocity is defined by $u^{\mu} = \frac{\mathrm{d}}{\mathrm{d}\tau}x^{\mu} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\,(ct, x(t), y(t), z(t))$, with the eigentime τ choosen so that $u^{\mu}u_{\mu} = c^2$. Hence we find

$$u'^{\nu} = (c,0,0,0)\,, \tag{5}$$

and

$$u^{\mu} = \frac{1}{\sqrt{1-(V/c)^2}}\,(c,0,0,V)\,. \tag{6}$$

d) Which energy $\hbar\omega$ is the photon observed to have? Express your answer by $\hbar\omega_0$, the angle θ, and V.

In the rest system of the object the photon is specified by the four-momentum

$$p'^{\nu} = \frac{\omega_0}{c}\,(1, \sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta')\,, \tag{7}$$

with ω_0 independent of direction. In our system it is specified by the four-momentum

$$p^{\mu} = \frac{\omega}{c}\,(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \tag{8}$$

with $\omega = \omega(\theta,\phi)$ depending on the direction of the photon.

One simple method to find the connection is to use the invariance of scalar products, implying that $u'_{\nu}p'^{\nu} = u_{\mu}p^{\mu}$,

$$u'_{\nu}p'^{\nu} = \omega_0 = u_{\mu}p^{\mu} = \frac{1-(V/c)\cos\theta}{\sqrt{1-(V/c)^2}}\,\omega.$$

I.e.,

$$\hbar\omega = \frac{\sqrt{1-(V/c)^2}}{1-(V/c)\cos\theta}\,\hbar\omega_0. \tag{9}$$

By writing $V = c\tanh\eta$ we find that the photon is blueshifted by a factor e^{η} for $\theta = 0$ (early detection times), redshifted by a factor $\mathrm{e}^{-\eta}$ for $\theta = \pi$ (late detection times), and unshifted when $\cos\theta = \tanh\eta/2$.

A seemingly different method is to use the transformation formula between four-vectors in the two frames (with $V = c\tanh\eta$),

$$\begin{pmatrix} p'^0 \\ p'^x \\ p'^y \\ p'^z \end{pmatrix} = \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \begin{pmatrix} p^0 \\ p^x \\ p^y \\ p^z \end{pmatrix}. \tag{10}$$

This gives

$$p'^0 = \cosh\eta\,p^0 - \sinh\eta\,p^z = \cosh\eta\,(1-\tanh\eta\,\cos\theta)\,p^0,$$

which is equivalent to (9).

Now assume instead that the object is a point charge Q, so that it is surrounded by a rotation symmetric electric field in the coordinate system where it is at rest (at the origin)

$$\boldsymbol{E}'(t',\boldsymbol{x}') = \frac{Q\,\boldsymbol{x}'}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3}, \quad \boldsymbol{B}'(t',\boldsymbol{x}') = \boldsymbol{0}. \tag{11}$$

This may also be expessed by the four-potential

$$A'^{\mu}(t',\boldsymbol{x}') = \frac{Q}{4\pi\varepsilon_0|\boldsymbol{x}'|}\,(1,\boldsymbol{0})\,. \tag{12}$$

e) Express the quantity $|\boldsymbol{x}'|$ in our coordinates $(t,\boldsymbol{x})$ (where the detectors are at rest).

Hint: Some general transformation formulae is included at the end of the problemset.

The two coordinate systems are related by the Lorentz transformation (with $\tanh\eta = V/c$),

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \begin{pmatrix} \cosh\eta & 0 & 0 & -\sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

so that

$$\begin{aligned} |\boldsymbol{x}'| &= \sqrt{x'^2+y'^2+z'^2} = \sqrt{x^2+y^2+(\cosh\eta\,x - \sinh\eta\,ct)^2} \\ &= \sqrt{x^2+y^2+\cosh^2\eta(z-Vt)^2}, \end{aligned} \tag{13}$$

where $\cosh\eta = 1/\sqrt{1-(V/c)^2}$ (this quantity is commonly called γ).

f) Calculate the field $\boldsymbol{E}(t,x,0,0)$ observed by the detectors at position $\boldsymbol{R}$.

You may choose to use the transformation formula for the electromagnetic field tensor, or assume that the four-potential transform as a vector, and next compute the $\boldsymbol{E}$-field from the transformed potential $A^{\nu}(t,\boldsymbol{x})$.

The transformation formula for the electromagnetic field tensor reads

$$F^{\mu\nu}(x) = \Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\,F'^{\alpha\beta}(x'), \tag{14}$$

here with $\Lambda^{\mu}{}_{\nu}$ the inverse of the transformation matrix used in (10). We find

$$E^x(t,\boldsymbol{x}) = F^{0x}(x) = \Lambda^0{}_0\,\Lambda^x_x\,F'^{0x}(x') = \cosh\eta\,F'^{0x}(x') = \frac{Q\cosh\eta\,x}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3},$$

since $\Lambda^x{}_x = 1$ is the only matrix element with an upper index x, and $x' = x$. For the same reasons,

$$E^y(t,\boldsymbol{x}) = F^{0y}(x) = \Lambda^0{}_0\,\Lambda^y_y\,F'^{0y}(x') = \cosh\eta\,F'^{0y}(x') = \frac{Q\cosh\eta\,y}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3}.$$

Finally

$$E^z(t,\boldsymbol{x}) = F^{0z}(x) = \left(\Lambda^0{}_0\,\Lambda^z_z - \Lambda^0{}_z\Lambda^z{}_0\right)\,F'^{0z}(x') = F'^{0z}(x') = \frac{Q\cosh\eta\,(z-Vt)}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3}.$$

Here we have used that $F'^{x0} = -F'^{0x}$, that $\left(\Lambda^0{}_0\,\Lambda^z_z - \Lambda^0{}_z\Lambda^z{}_0\right) = \cosh^2\eta - \sinh^2\eta = 1$, and finally that $z' = \cosh\eta z - \sinh\eta\,ct = \cosh\eta\,(z-Vt)$.

Alternatively we could first transform the four-potential according to the formula

$$A^{\mu}(x) = \Lambda^{\mu}{}_{\nu}\,A'^{\nu}(x'), \tag{15}$$

with $\Lambda^{\mu}{}_{\nu}$ the inverse of the transformation matrix used in (10). This gives

$$A^{\mu}(x) = \frac{Q\,(\cosh\eta,0,0,\sinh\eta)}{4\pi\varepsilon_0\,|\boldsymbol{x}'|}, \tag{16}$$

from which we find, using $|\boldsymbol{x}'| = \sqrt{x^2 + y^2 + \cosh^2\eta\,(z - Vt)^2}$,

$$\begin{aligned}
E^x &= -\left(\frac{\partial A^x}{c\,\partial t} + \frac{\partial A^0}{\partial x}\right) = \frac{Q\cosh\eta\,x}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3},\\
E^y &= -\left(\frac{\partial A^y}{c\,\partial t} + \frac{\partial A^0}{\partial y}\right) = \frac{Q\cosh\eta\,y}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3},\\
E^z &= -\left(\frac{\partial A^z}{c\,\partial t} + \frac{\partial A^0}{\partial z}\right) = \frac{Q\cosh\eta\,(z - Vt)}{4\pi\varepsilon_0\,|\boldsymbol{x}'|^3}.
\end{aligned}$$

The computation of E^x and E^y is very simple since $A^x = A^y = 0$. For E^z the numerator combine as

$$\left(-\sinh\eta\,\cosh^2\eta\,\tanh\eta + \cosh\eta\,\cosh^2\eta\right)\,(z - Vt) = \cosh\eta\,(z - Vt),$$

where we have used that $V/c = \tanh\eta$.

By either method we find

$$\begin{aligned}
E^x(t, x, 0, 0) &= \frac{Qx\,\cosh\eta}{4\pi\varepsilon_0\left[\,x^2 + (\cosh\eta\,Vt)^2\,\right]^{3/2}},\\
E^y(t, x, 0, 0) &= 0,\\
E^z(t, x, , 0, 0) &= \frac{-QVt\,\cosh\eta}{4\pi\varepsilon_0\left[\,x^2 + (\cosh\eta\,Vt)^2\,\right]^{3/2}}.
\end{aligned} \tag{17}$$

g) Compare the observed direction of the field, $\cot\vartheta \equiv E^z/E^x$, with the observed direction $\cot\theta$ of the photons in point **b)**.

We find $\cot\vartheta = -Vt/x$ compared to $\cot\theta = -Vt_0/x$. I.e., the electric field points away from the *current* (in our coordinates) position of the object, while the photon momentum points away from the position of the object at the *time of emission*.

Problem 2. The Friedmann-Lemaître-Robertson-Walker universe

The Friedmann-Lemaître-Robertson-Walker metric is defined by the line element

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right), \tag{18}$$

where we units where the speed of light $c = 1$, and where $k \in \{-1, 0, 1\}$.

a) Write down (i) the metric tensor $g_{\mu\nu}$, and (ii) the inverse metric tensor $g^{\mu\nu}$ for the universe defined by the line element (18). Assume that the metric tensor has signature $(-, +, +, +)$.

We compare equation (18) with the general expression

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

to find

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & \frac{a(t)^2}{1-kr^2} & 0 & 0\\ 0 & 0 & a(t)^2\,r^2 & 0\\ 0 & 0 & 0 & a(t)^2\,r^2\sin^2\theta \end{pmatrix}. \tag{19}$$

The inverse metric becomes

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & \frac{1-kr^2}{a(t)^2} & 0 & 0\\ 0 & 0 & \frac{1}{a(t)^2r^2} & 0\\ 0 & 0 & 0 & \frac{1}{a(t)^2r^2\sin^2\theta} \end{pmatrix}. \tag{20}$$

b) Calculate the integration measure $\sqrt{-g}$ for the universe defined by the line element (18).

Computing the determinant of (19) gives

$$\sqrt{-g} = \frac{a(t)^3\, r^2}{\sqrt{1-kr^2}}\,\sin\theta. \tag{21}$$

c) Explain in qualitative terms what is meant by (i) covariant derivative, (ii) connection coefficients, (iii) Riemann tensor, (iv) Ricci tensor, (v) Einstein tensor, and (vi) scalar curvature. Explain briefly how you would compute these quantities from the line element (18). You need not perform any explicit computations here, but you should indicate the index structure of the quantities and relations involved.

(i) The covariant derivative $D_\mu = \partial_\mu + \mathbf{\Gamma}_\mu$ is the correct differentiation operator on vectors (and tensors), taking into account that components of such quantities are with respect to a basis which may be changing with position.

(ii) The components of the matrices $\mathbf{\Gamma}_\mu$ makes up the connection coeffiients $\Gamma^\alpha{}_{\beta\mu}$. For a metric connection they can f.i. be found from the geodesic equations following from the metric $g_{\mu\nu}$, by applying Hamiltons principle to the action $S = -\frac{1}{2}\int \mathrm{d}\tau\, g_{\mu\nu}\,\dot{x}^\mu\,\dot{x}^\nu$ and comparing with the general form

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\mu}\,\dot{x}^\beta\,\dot{x}^\mu = 0. \tag{22}$$

Or they may be computed from the general expression

$$\Gamma^\alpha{}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\gamma\beta,\mu} + g_{\gamma\mu,\beta} - g_{\beta\mu,\gamma}\right). \tag{23}$$

(iii) The Riemann tensor con be defined as the commutator $[D_\mu, D_\nu] \equiv \boldsymbol{R}_{\mu\nu}$. For each $\mu\nu$-combination this is a matrix with components $R^\alpha{}_{\beta\mu\nu}$. In four space-time dimensions there are alltogether $4^4 = 256$ components, out of which 20 are algebraically independent.

(iv) The Ricci tensor is obtained contraction of the Riemann tensor. Usually

$$R_{\beta\nu} = R^\alpha{}_{\beta\alpha\nu}, \tag{24}$$

but sometimes with the opposite sign.

(v) The Einstein tensor is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, g^{\alpha\beta} R_{\alpha\beta}. \tag{25}$$

It is distinguished by automatically being covariantly conserved,

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{26}$$

(vi) The scalar curvature is defined as the trace of the Ricci tensor,

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{27}$$

As a scalar it is a natural candidate for a Lagrangian for the gravity field.

d) Assume that the matter content of this universe can be modelled by a scalar field φ and the associated action

$$S_{\text{matter}} = -\int \mathrm{d}^4x\,\sqrt{-g}\,\left(\frac{1}{2} g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi + V(\varphi)\right), \tag{28}$$

where V is some differentiable function of its argument. Assume that the field φ only depends on time, $\varphi = \varphi(t)$.

Simplify the action S_{matter} for this case, and find the Euler-Lagrange equation for the field φ.

The action can be written as,

$$S_{\mathrm{matter}} = \int \mathrm{d}t\, a^3 \left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right] \int_{\mathcal{V}} \frac{r^2 \mathrm{d}r\, \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi}{\sqrt{1-kr^2}}, \tag{29}$$

where the intergral over spatial coordinates may be treated as a constant $\mathcal{V}$.

The Euler-Lagrange equations becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3\,\dot{\varphi}\right) = -a^3\,V'(\varphi),$$

which simplifies to

$$\ddot{\varphi} + \frac{3\dot{a}}{a}\,\dot{\varphi} + V'(\varphi) = 0. \tag{30}$$

I.e., the expansion of the universe induces a "friction force" in the equations of motion.

e) The Hilbert-Einstein action for the gravity part of the action is

$$S_{\mathrm{HE}} = \frac{1}{16\pi G_N}\int \mathrm{d}^4x\,\sqrt{-g}\,R, \tag{31}$$

where G_N is Newtons constant of gravity, and R is the scalar curvature. For the length element (18) the scalar curvature is

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2 + k\right). \tag{32}$$

Simplify the action S_{HE} to the case that all dynamic quantities only depend on t (like you did for S_{matter} in the previous point), and use Hamiltons principle for the total action, $S_{\mathrm{total}} = S_{\mathrm{HE}} + S_{\mathrm{matter}}$, to find the equation of motion for φ.

We can write

$$S_{\mathrm{HE}} = \frac{3}{8\pi\,G_N}\int \mathrm{d}t\,\left(a^2\ddot{a} + a\dot{a}^2 + ka\right)\int_{\mathcal{V}} \frac{r^2 \mathrm{d}r\, \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi}{\sqrt{1-kr^2}}. \tag{33}$$

We may perform a partial integration in the first term to bring this into a more familiar form involving at most first order derivatives,

$$\bar{S}_{\mathrm{HE}} = \frac{3}{8\pi\,G_N}\int \mathrm{d}t\,\left(-a\dot{a}^2 + ka\right)\int_{\mathcal{V}} \frac{r^2 \mathrm{d}r\, \sin\theta \mathrm{d}\theta\, \mathrm{d}\phi}{\sqrt{1-kr^2}},$$

but it is also possible to apply Hamiltons principle to the original action. We obtain a total action

$$\bar{S}_{\mathrm{total}} = \int \mathrm{d}t\,\left\{\left(-a\dot{a}^2 + ka\right) + \frac{8\pi G_N}{3}\,a^3\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)\right\}\left(\frac{3\,\mathcal{V}}{8\pi G_N}\right), \tag{34}$$

form which we find the Euler-Lagrange equation for a,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-2a\dot{a}\right) = \left\{\left(-\dot{a}^2 + k\right) + 8\pi G_N\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right)\right\},$$

or simplified as

$$\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = -8\pi G_N\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right). \tag{35}$$

f) The action S_{total} is invariant under time translations, $\varphi(t) \to \varphi(t+\varepsilon)$ and $a(t) \to a(t+\varepsilon)$. Use Nöthers theorem to find the corresponding conserved quantity.

In this case Nöthers theorem says that the conserved quantity is

$$E = \frac{\partial L}{\partial \dot{a}}\delta a + \frac{\partial L}{\partial \dot{\varphi}}\delta\varphi - L,$$

with $\delta a = \dot{a}$, $\delta\varphi = \dot{\varphi}$, and L given by the integrand in (34), except that we will ignore the constant $3\,\mathcal{V}/(8\pi G_N)$. With $\partial L/\partial\dot{a} = -2a\dot{a}$ and $\partial L/\partial\dot{\varphi} = a^3\dot{\varphi}$ we find

$$E = -\left(a\dot{a}^2 + ka\right) + \frac{8\pi G_N}{3}a^3\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right). \tag{36}$$

Remark: The analysis above do not lead to the complete set of Einstein equations. A full analysis gives the additional constraint that $E = 0$, equivalent to the condition

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3}G_N\left(\frac{1}{2}\dot{\varphi}^2 + V(\phi)\right), \tag{37}$$

which is known as the *first Friedmann equation*.