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Løsningsforslag til eksamen i FY3452 GRAVITASJON OG KOSMOLOGI

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Dette løsningsforslaget er på 4 sider.

Oppgave 1. Bevegelse utenfor et roterende legeme

Til laveste ikke-trivielle orden i r_M/r er linjeelementet utenfor et roterende legeme med masse M og dreieimpuls J av formen

$$c^2\,\mathrm{d}\tau^2 = \left(1-\frac{r_M}{r}\right)c^2\mathrm{d}t^2 + 2K_J\frac{r_M^2}{r^2}\sin^2\theta\,cr\,\mathrm{d}t\,\mathrm{d}\phi - \left(1+\frac{r_M}{r}\right)\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2\right). \tag{1}$$

Her er

$$r_M = \frac{2G_N M}{c^2}, \quad K_J = \frac{J}{Mc\,r_M}, \tag{2}$$

der G_N er Newton's konstant. Bevegelsen til en punktpartikkel utenfor dette legemet er bestemt av Lagrangefunksjonen

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\,\dot{x}^\nu, \tag{3}$$

via Hamiltons prinsipp. Her betyr $\dot{}$ derivasjon med hensyn til egentid τ. Du kan velge å bruke enheter der $c=1$.

a) Lagrangefunksjonen L avhenger ikke eksplisitt av t. Hvilken konservert størrelse gir dette opphav til?

The corresponding conserved quantity is

$$\epsilon \equiv \frac{\partial L}{\partial \dot{t}} = \left(1-\frac{r_M}{r}\right)\dot{t} + K_J\frac{r_M^2}{r^2}\sin^2\theta\,r\dot{\phi}. \tag{4}$$

b) Lagrangefunksjonen L avhenger ikke eksplisitt av ϕ. Hvilken konservert størrelse gir dette opphav til?

The corresponding conserved quantity is

$$\ell \equiv \frac{\partial L}{\partial \dot{\phi}} = K_J\frac{r_M^2}{r^2}\sin^2\theta\,r\dot{t} - r^2\sin^2\theta\,\dot{\phi}. \tag{5}$$

c) Lagrangefunksjonen L avhenger ikke eksplisitt av τ. Hvilken konservert størrelse gir dette opphav til?

The corresponding conserved quantity is

$$h \equiv \frac{\partial L}{\partial \dot{t}}\dot{t} + \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial \dot{\phi}}\dot{\phi} - L = 2L - L = L, \tag{6}$$

since L is homogeneous of degree two in the four-velocity $\dot{x}^\mu$.

d) Anta at $\theta = \frac{1}{2}\pi$, $\dot{\theta} = 0$, dvs. bevegelse i ekvatorplanet, er en løsning av bevegelsesligningene. Sett derfor $\sin^2\theta = 1$, $\dot{\theta} = 0$, og finn bevegelsesligningen for $r(\tau)$.

The Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}. \tag{7}$$

The quantities $\partial L/\partial \dot{r}$ and $\partial L/\partial r$ can be computed directly from the original expression for L. However, these expressions depend on $\dot{t}$ and $\dot{\phi}$ which should be eliminated by use of (4) and (5) before proceeding.

Oppgave 2. Estimat av størrelsesorden

Bruk din generelle kunnskap om fysiske fenomener og fysiske sammenhenger til å anslå størrelsene nedenfor. Forklar hvordan du kom fram til anslagene.

a) Parameteren $r_{M_\oplus}/r_\oplus$, der $M_\oplus$ er massen til jorda og $r_\oplus$ er jordas radius.

The acceleration of gravity ($g \approx 9.81$ m/s^2), the original definition the meter ($\frac{1}{2} r_\oplus = 10^4$ km), and the speed of light ($c = 3\,10^8$ m/s) allow us to find $G_N M_\oplus$.

b) Parameteren $r_{M_\odot}/r_\odot$, der $M_\odot$ er massen til sola og $r_\odot$ er solas radius.

The length of the year ($365.25 \times 24 \times 60 \times 60$ seconds), and distance to the sun (150 million kilometers or 8 light-minutes) allow us to find $G_N M_\odot$. This next give us the ratio $M_\odot/M_\oplus$. Since the density of normal matter does not vary much (the sun has a somewhat lower density than the earth) this further allow us to estimate $r_\odot$. This quantity may also be estimated from its apparent size on the sky.

c) Parameteren $K_{J_\oplus} = \frac{J_\oplus}{M_\oplus c\, r_\oplus}$ for jorda, der $J_\oplus$ er dreieimpulsen til jorda.

The angular momentum $J_\oplus = \omega_\oplus I_\oplus$, where $\omega_\oplus$ is the rotation speed ($\approx 2\pi/24$ hours) and $I_\oplus \propto M_\oplus r_\oplus^2$. The value of $M_\oplus$ drops out in the combination.

d) Parameteren $K_{J_\odot} = \frac{J_\odot}{M_\odot c\, r_\odot}$ for sola, der $J_\odot$ er dreieimpulsen til sola.

The angular momentum $J_\odot = \omega_\odot I_\odot$, where $\omega_\odot$ is the rotation speed ($\approx 2\pi/28$ days) and $I_\odot \propto M_\odot r_\odot^2$. The value of $M_\odot$ drops out in the combination.

Oppgave 3. Einstein's gravitasjonsteori til laveste orden

I denne oppgaven skal du se litt på Einstein gravitasjonsteori til første orden i avviket fra flatt rom. Dvs. at vi skriver linjeelementet på formen

$$c^2\,\mathrm{d}\tau^2 = \{\eta_{\mu\nu} + \varepsilon\, h_{\mu\nu}(x)\}\,\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{8}$$

der $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$, og bare regner til første orden i parameteren ε. Dette er tilstrekkelig til å relativt enkelt kunne finne linjeelementer som f.eks. (1).

a) Anta at vi gjør en (liten) transformasjon av koordinater,

$$x^\mu = \tilde{x}^\mu + \varepsilon\,\Lambda^\mu(\tilde{x}), \tag{9}$$

og regn ut den tilhørende transformasjonen,

$$h_{\mu\nu}(x) \to \bar{h}_{\mu\nu}(\tilde{x}). \tag{10}$$

We insert

$$\mathrm{d}x^\mu = \mathrm{d}\bar{x}^\mu + \varepsilon\left(\frac{\partial \Lambda^\mu}{\partial \tilde{x}^\lambda}\right)\mathrm{d}\tilde{x}^\lambda,$$

to find

$$c^2\,\mathrm{d}\tau^2 = \left\{\eta_{\mu\nu} + \varepsilon\,(h_{\mu\nu} + \Lambda_{\mu,\nu} + \Lambda_{\nu,\mu}) + \mathcal{O}(\varepsilon^2)\right\}\mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}^\nu, \tag{11}$$

where $\Lambda_{\mu,\nu} \equiv \left(\frac{\partial\Lambda_\mu}{\partial\bar{x}^\nu}\right)$, and all terms on the right hand side is evaluated at $\bar{x}$. The difference between $h_{\mu\nu}(x)$ and $h_{\mu\nu}(\bar{x})$ is a contribution to the $\mathcal{O}(\varepsilon^2)$-terms. Hence we find

$$\tilde{h}_{\mu\nu}(\tilde{x}) = h_{\mu\nu}(\tilde{x}) + \Lambda_{\mu,\nu}(\tilde{x}) + \Lambda_{\nu,\mu}(\tilde{x}). \tag{12}$$

b) Vis at det er mulig å velge $\Lambda^\mu(\tilde{x})$ slik at

$$V_\nu(\tilde{h}) \equiv \partial_\mu \left(\tilde{h}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu\, \tilde{h}^\lambda{}_\lambda\right) = 0. \tag{13}$$

I det følgende kan du anta at denne betingelsen allerede er oppfylt for $h_{\mu\nu}$, dvs. at $V_\nu(h) = 0$.

The requirement reduces to the equation

$$\partial^\mu\partial_\mu\Lambda_\nu + V_\nu \equiv \Box\Lambda_\nu + V_\nu = 0, \tag{14}$$

where $V_\nu = \partial^\mu\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\lambda{}_\lambda\right)$. We may in principle always solve this equation for Λ_ν.

c) Bestem konneksjonskoeffisientene $\Gamma^\mu{}_{\nu\lambda}$ til første orden i ε.

We find

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\left(h^\mu{}_{\nu,\lambda} + h^\mu{}_{\lambda,\nu} - \partial^\mu h_{\nu\lambda}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{15}$$

d) Vis at Riemann-tensoren kan uttrykkes på formen

$$R_{\mu\nu\lambda\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\lambda} + h_{\nu\lambda,\mu\sigma} - h_{\nu\sigma\mu\lambda} - h_{\mu\lambda,\nu\sigma}\right)\varepsilon + \mathcal{O}(\varepsilon^2). \tag{16}$$

In matrix form the Riemann tensor is defined by the expression

$$\boldsymbol{R}_{\lambda\sigma} = \partial_\lambda\boldsymbol{\Gamma}_\sigma - \partial_\sigma\boldsymbol{\Gamma}_\lambda + \mathcal{O}(\varepsilon^2),$$

where

$$(\boldsymbol{R}_{\lambda\sigma})^\mu{}_\nu \equiv R^\mu{}_{\nu\lambda\sigma}, \quad (\boldsymbol{\Gamma}_\lambda)^\mu{}_\nu \equiv \Gamma^\mu{}_{\nu\lambda}.$$

Hence we find to order ε,

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda\Gamma^\mu{}_{\nu\sigma} - \partial_\lambda\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}\left(h^\mu{}_{\sigma,\nu\lambda} + h_{\nu\lambda,}{}^\mu{}_\sigma - h_{\nu\sigma,}{}^\mu{}_\lambda - h^\mu{}_{\lambda,\nu\sigma}\right)\varepsilon,$$

and obtain (16) after lowering the μ-index.

e) Hver av de fire indeksene til $R_{\mu\nu\lambda\sigma}$ kan ta fire verdier $(0,1,2,3)$. Hvor mange *uavhengige* komponenter har $R_{\mu\nu\lambda\sigma}$ for en generell symmetrisk $h_{\mu\nu}$?

We see from (16) that $R_{\mu\nu\lambda\sigma}$ is antisymmetric under the interchange $\mu \rightleftarrows \nu$ (keeping λ,σ fixed) and $\lambda \rightleftarrows \sigma$ (keeping μ,ν fixed), and symmetric under the interchange $(\mu,\nu) \rightleftarrows (\lambda,\sigma)$. For computation of independent components the pairs (μ,ν) and (λ,ν) may therefore be restricted to $\frac{1}{2}\times 4\times 3 = 6$ values each,

$$I, J = (0,1), (0,2), (0,3), (1,2), (1,3), (2,3),$$

and the Riemann tensor viewed as a symmetric 6×6 matrix $R_{I,J}$. Such matrices has $\frac{1}{2}\times 6\times 7 = 21$ independent elements. This is considered a sufficient answer.

Extra bonus to those who knows that there is one more independent restriction,

$$R_{0123} + R_{0231} + R_{0312} = 0, \tag{17}$$

leaving *20 independent components.*

f) Beregn Ricci-tensoren $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$. Bruk betingelsen $V_\nu(h) = 0$ til å forenkle uttrykket. Beregn Einstein-tensoren $G_{\mu\nu} = R_{\mu\nu} - \eta_{\mu\nu}\, R^{\lambda}{}_{\lambda}$ med den samme betingelsen.

By careful rewriting of indices, and contraction, we find from (16),

$$R_{\mu\nu} = \frac{1}{2}\left[-\Box h_{\mu\nu} + \partial_\mu V_\nu(h) + \partial_\nu V_\mu(h)\right]\varepsilon. \tag{18}$$

The last two terms vanishes when we use the condition $V_\nu(h) = 0$. Under the same condition we obtain

$$G_{\mu\nu} = -\frac{1}{2}\varepsilon\,\Box\,\bar{h}_{\mu\nu} + \mathcal{O}(\varepsilon^2), \tag{19}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{\lambda}{}_{\lambda} \tag{20}$$

is called the *trace-reversed* metric. Note that the condition $V_\nu(h) = 0$ simplifies to $\partial_\mu \bar{h}^{\mu}{}_{\nu} = 0$.

Some expressions which *may* be of use

Euler-Lagrange equations

The Euler-Lagrange equations for a field theory described by the Lagrangian $\mathcal{L} = \mathcal{L}(\varphi_a, \partial_\mu \varphi_a, x)$ are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi_a}. \tag{21}$$

The corresponding equations for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.

Nöther's theorem

Assume the action is invariant under the continuous transformations $\varphi_a \to \varphi_a + \varepsilon\, \delta\varphi_a + \mathcal{O}(\varepsilon^2)$, more precisely that $\mathcal{L} \to \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu + \mathcal{O}(\epsilon^2)$ under this transformation. Then there is an associated conserved current,

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a - \Lambda^\mu. \tag{22}$$

I.e., $\partial_\mu J^\mu = 0$. The corresponding expression for point particle mechanics is obtained by restricting ∂_μ to only a time derivative $\mathrm{d}/\mathrm{d}t$.