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Solution to the exam in FY3466 QUANTUM FIELD THEORY II

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This solution consists of 7 pages.

Problem 1. Model for fermions and bosons

In this problem we shall investigate a much simplified version of the Glashow-Weinberg-Salam (GWS) model of electroweak interactions. Consider first the Lagrangian density

$$\mathcal{L} = \mathrm{i}L^\dagger \bar{\sigma}^\nu \,\partial_\nu L + \mathrm{i}R^\dagger \sigma^\nu \,\partial_\nu R + (\partial^\nu \varphi^*)\,(\partial_\nu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{4}(\varphi^* \varphi)^2, \tag{1}$$

where μ^2 and λ are positive parameters. In equation (1) L and R are two-component spinor fields, $\sigma^\nu = (I, \boldsymbol{\sigma})$ and $\bar{\sigma}^\nu = (I, -\boldsymbol{\sigma})$ where $\boldsymbol{\sigma}$ are the Pauli matrices.

a) What is the classical groundstate (minimum energy state) of this model? I.e., what are the values of the fields L, R and φ in this state, and what is the corresponding energy density?

A groundstate is obtained for $L = R = 0$, and

$$\varphi^* \varphi = \frac{2\mu^2}{\lambda}. \tag{2}$$

The phase of φ is arbitrary. The minimum energy density becomes

$$\mathcal{H}_{\min} = -\frac{\mu^4}{\lambda}. \tag{3}$$

Remark: A canonical analysis of the model shows that

$$\Pi_L = \mathrm{i}L^\dagger, \quad \Pi_R = \mathrm{i}R^\dagger, \quad \Pi_\varphi = \dot{\varphi}^*, \quad \Pi_{\varphi^*} = \dot{\varphi}, \tag{4}$$

leading to an energy (Hamiltonian) density

$$\mathcal{H} = \mathrm{i}L^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} L - \mathrm{i}R^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} R + \dot{\varphi}^* \dot{\varphi} + \boldsymbol{\nabla}\varphi^* \cdot \boldsymbol{\nabla}\varphi - \mu^2 \varphi^* \varphi + \frac{1}{4}\lambda(\varphi^* \varphi)^2. \tag{5}$$

For entirely classical (i.e. commuting) fields, L and R, this expression has no lower bound. This can be seen by f.i. taking

$$L = \mathrm{e}^{\mathrm{i}kz} \begin{pmatrix} 0 \\ 1 \end{pmatrix} L_0, \quad R = \mathrm{e}^{\mathrm{i}kz} \begin{pmatrix} 1 \\ 0 \end{pmatrix} R_0,$$

with $k > 0$. Then $\mathcal{H} \to -\infty$ when $L_0 \to \infty$ and/or $R_0 \to \infty$. However, as fermion fields the "classical" limit ($\hbar \to 0$) of L_0 and R_0 are Grassmann variables giving no sensible contribution to a classical energy density. Hence, it is standard procedure to neglect the fermions when searching for classical ground states.

b) What are, to lowest order in the parameter λ, the masses of the particles (the quantized fluctuations around the ground state) in this model?

The masses of the (anti-)particles created by the L and R fields are zero. For the bosonic field φ we choose the phase so that its vacum expectation value is

$$\varphi_0 = \sqrt{\frac{2\mu^2}{\lambda}}, \tag{6}$$

and write

$$\varphi(x) = \varphi_0 + \frac{1}{\sqrt{2}}\left[\phi_1(x) + \mathrm{i}\phi_2(x)\right]. \tag{7}$$

The Lagrangian density involving the ϕ-fields the becomes, to quadratic order,

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}(2\mu^2)\phi_1^2 - \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2). \tag{8}$$

Thus, the mass of the bosons are

$$m_1 = \sqrt{2}\mu, \quad m_2 = 0. \tag{9}$$

c) This model is invariant under three independent global phase transformations, $U(1)\times U(1)\times U(1)$? Which phase transformations?

One possible choice is

$$L \to \mathrm{e}^{\mathrm{i}\alpha} L, \quad R \to \mathrm{e}^{\mathrm{i}\beta} R, \quad \varphi \to \mathrm{e}^{\mathrm{i}\gamma}\varphi. \tag{10}$$

We now add an interaction term to the Lagrangian density,

$$\Delta\mathcal{L} = -\lambda_m \left(\varphi L^\dagger R + \varphi^* R^\dagger L\right) \tag{11}$$

d) This model now becomes invariant under *two* independent global phase transformations, $U(1)\times U(1)$. Show how these two transformations act on the three fields.

We must now assure that the contribution (11) remains invariant. One possible choice is

$$L \to \mathrm{e}^{\mathrm{i}\alpha} L, \quad R \to \mathrm{e}^{\mathrm{i}\beta} R, \quad \varphi \to \mathrm{e}^{\mathrm{i}(\alpha-\beta)}\varphi. \tag{12}$$

e) What is the mass of the fermion (to lowest order in λ and λ_m) in the modified model?

The Lagrangian density involving only L, R and φ_0 becomes

$$\mathcal{L}_{\text{fermion}} = \mathrm{i}L^\dagger\bar{\sigma}^\nu\,\partial_\nu L + \mathrm{i}R^\dagger\sigma^\nu\,\partial_\nu R - \lambda_m\varphi_0\left(L^\dagger R + R^\dagger L\right), \tag{13}$$

leading to the equation of motion

$$\begin{pmatrix} \mathrm{i}\bar{\sigma}^\nu\,\partial_\nu & -\lambda_m\varphi_0 \\ -\lambda_m\varphi_0 & \mathrm{i}\sigma^\nu\,\partial_\nu \end{pmatrix}\begin{pmatrix} L \\ R \end{pmatrix} = 0. \tag{14}$$

This is the Dirac equation in the Weyl representation, with mass term

$$m_{\mathrm{f}} = \lambda_m\,\varphi_0 = \lambda_m\sqrt{\frac{2\mu^2}{\lambda}}. \tag{15}$$

Remark: It may not be common to walk around remembering the form of the Dirac equation in the Weyl representation. For those who don't one straightforward approach is to eliminate R from the first equation in (14),

$$R = \frac{1}{\lambda_m \varphi_0} (\mathrm{i}\bar{\sigma}^\nu \partial_\nu) L.$$

This inserted into the second (and multiplied by $\lambda_m \varphi_0$) gives

$$(\mathrm{i}\sigma^\nu \partial_\nu)(\mathrm{i}\bar{\sigma}^\sigma \partial_\sigma) L = (\lambda_m \varphi_0)^2 L. \tag{16}$$

Now

$$(\mathrm{i}\sigma^\nu \partial_\nu)(\mathrm{i}\bar{\sigma}^\sigma \partial_\sigma) = -(\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla})(\partial_0 - \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}) = -\partial_0^2 + \boldsymbol{\nabla}^2 = -\Box.$$

Thus, equation (16) satisfied by L is just the Klein-Gordon equation,

$$\left(\Box + m_{\mathrm{f}}^2\right) L = 0, \qquad \text{with } m_{\mathrm{f}} = \lambda_m \varphi_0.$$

This shows that the fermions have mass m_{f}.

f) The two global phase transformations can be made local by introducing two gaugefields, B_ν og W_ν, and the associated covariant derivatives. Find a consistent set of covariant derivatives ($D_\nu^{(L)}$, $D_\nu^{(R)}$, $D_\nu^{(\varphi)}$), and write down the new Lagrangian density for the model.

You may assume that the covariant derivatives don't contain any interaction parameters, but that the kinetic terms for the gauge fields are

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4g^2} W_{\nu\sigma} W^{\nu\sigma} - \frac{1}{4g'^2} B_{\nu\sigma} B^{\nu\sigma}, \tag{17}$$

where $B_{\nu\sigma} = \partial_\nu B_\sigma - \partial_\sigma B_\nu$ and $W_{\nu\sigma} = \partial_\nu W_\sigma - \partial_\sigma W_\nu$.

One possible solution is

$$D_\nu^{(L)} = \partial_\nu + \mathrm{i}B_\nu, \tag{18}$$

$$D_\nu^{(R)} = \partial_\nu + \mathrm{i}W_\nu, \tag{19}$$

$$D_\nu^{(\varphi)} = \partial_\nu + \mathrm{i}B_\nu - \mathrm{i}W_\nu. \tag{20}$$

The transformation rules (12), with $\alpha = \alpha(x)$ and $\beta = \beta(x)$, must then be augmented with

$$B_\nu \to B'_\nu = B_\nu + \partial_\nu \alpha, \tag{21}$$

$$W_\nu \to W'_\nu = W_\nu + \partial_\nu \beta. \tag{22}$$

The Lagrangian for the complete model becomes

$$\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathrm{i}L^\dagger \bar{\sigma}^\nu D_\nu^{(L)} L + \mathrm{i}R^\dagger \sigma^\nu D_\nu^{(R)} R + (D^{(\varphi)\nu}\varphi)^* (D_\nu^{(\varphi)}\varphi) \tag{23}$$

$$+ \mu^2 \varphi^* \varphi - \frac{\lambda}{4}(\varphi^*\varphi)^2 - \lambda_m \left(\varphi L^\dagger R + \varphi^* R^\dagger L\right). \tag{24}$$

g) A linear combination of the gauge fields B_ν and W_ν gets quanta with mass M_Z in this model. Which linear combination? Find M_Z^2 expressed by the parameters μ^2, λ, g and g'.

To obtain the standard normalization of the kinetic terms for the gauge fields, we introduce rescaled fields so that

$$W_\nu = g\bar{W}_\nu, \quad B_\nu = g'\bar{B}_\nu. \tag{25}$$

The quadratic part of the gauge field Lagrangian then becomes

$$\mathcal{L}_2 = -\frac{1}{4}\left(\partial_\nu \bar{B}_\sigma - \partial_\sigma \bar{B}_\nu\right)\left(\partial^\nu \bar{B}^\sigma - \partial^\sigma \bar{B}^\nu\right) - \frac{1}{4}\left(\partial_\nu \bar{W}_\sigma - \partial_\sigma \bar{W}_\nu\right)\left(\partial^\nu \bar{W}^\sigma - \partial^\sigma \bar{W}^\nu\right) + \varphi_0^2\left(g'\bar{B}_\nu - g\bar{W}_\nu\right)\left(g'\bar{B}^\nu - g\bar{W}^\nu\right),$$

We write $g = \sqrt{g^2+g'^2}\cos\theta$ and $g' = \sqrt{g^2+g'^2}\sin\theta$, and introduce new fields

$$Z_\nu = -\sin\theta\, B_\nu + \cos\theta\, W_\nu, \quad A_\nu = \cos\theta\, B_\nu + \sin\theta\, W_\nu. \tag{26}$$

This is an orthogonal transformation, which means that the kinetic terms preserve their diagonal form. The last term becomes $\varphi_0^2\left(g^2+g'^2\right) Z_\nu Z^\nu$, giving

$$\mathcal{L}_2 = -\frac{1}{4}\left(\partial_\nu A_\sigma - \partial_\sigma A_\nu\right)\left(\partial^\nu A^\sigma - \partial^\sigma A^\nu\right) - \frac{1}{4}\left(\partial_\nu Z_\sigma - \partial_\sigma Z_\nu\right)\left(\partial^\nu Z^\sigma - \partial^\sigma Z^\nu\right) + \frac{1}{2}M_Z^2\, Z_\nu Z^\nu, \tag{27}$$

with M_Z the mass of the Z-boson,

$$M_Z = \varphi_0\sqrt{2\left(g^2+g'^2\right)} = 2\mu\sqrt{\frac{g^2+g'^2}{\lambda}}. \tag{28}$$

Remark: Note that the detailed expressions above depends on the specific form chosen for the covariant derivate $D_\nu^{(\varphi)}$.

Problem 2. Concepts in Quantum Field Theory

Give a short qualitative description of the following concepts

a) Ward identity.

Ward identities are relations (which must be) satisfied by amplitudes in a gauge theory like QED. Typically, if we have an amplitude $\mathcal{M}_{\nu\cdots}$ describing the emission or absorption of a photon with four-momentum k, then we must have $k^\nu\mathcal{M}_{\nu\cdots} = 0$. This is required for the amplitude to be independent of a change of the polarization vector $e^\nu \to e^\nu + k^\nu$.

b) Dimensional regularization.

Dimensional regularization is a convenient and gauge invariant method to evaluate Feynman diagram loop integrals which are divergent in $d = 4$ space-time dimensions. It is based on the fact that such integrals typically have can be analytically continued to non-integer (even complex) dimensions. By f.i. writing $d = 4 + 2\epsilon$ the divergences manifest themselves as poles at $\epsilon = 0$ (and other integer values).

c) Wick rotation (in Feynman diagram integrations)

A Feynman diagram loop integrand typically involve factors like

$$\frac{1}{\left(k^2 - a^2 + \mathrm{i}\epsilon\right)^n},$$

where $k^2 = k_0^2 - \boldsymbol{k}^2$, with integration over $\mathrm{d}^d k$. The integrand thus have pole singularities near the k_0 integration axis, which however can be rotated to the imaginary axis without encountering singularites. I.e. we change the k_0 integration contour to

$$k_0 = \mathrm{e}^{\mathrm{i}\alpha} k_{\mathrm{E}}$$

with k_E integrated along the real axis, and increase α from 0 to $\frac{\pi}{2}$. This is a *Wick rotation* (there is a correspondig rotation of real time to imaginary time).

d) The Cabibbo-Kobayashi-Maskawa matrix.

In the GSW model of electroweak interactions (with 3 quark generations) the charged weak interaction connects the upper and lower components of lefthanded quark doublets. However, the fields in each doublet are not the same as those forming mass eigenstates. This leads to a unitary 3×3 *mixing matrix* parametrizing the relative transition amplitudes between upper and lower components of the various generations. This is the *Cabibbo-Kobayashi-Maskawa matrix*.

e) Neutrino oscillations.

The neutrinos created from the various charged leptons (e, μ, τ) in a charged weak interaction process are not exact mass eigenstates, but a linear superposition of such. This means that f.i. an electron neutrino will *oscillate* other types of neutrinos after sufficiently long time (or propagation distance).

f) Spontaneous symmetry breakdown.

The ground state of f.i. a quantum field theory with some symmetry may not necessarily be an eigenstate of that symmetry. This is called *spontaneous symmetry breakdown*.

g) Goldstone boson.

The consequence of the spontaneous breakdown of a continous symmetry is that there (usually) will exist gapless excitations, corresponding to massless scalar particles in relativistic theories. These are the Goldstone bosons.

h) Landau, Feynman, and Yennie gauge.

Calculating the Feynman propagator for a gauge field A^ν with the gauge fixing condition $\partial_\nu A^\nu(x) = \omega(x)$, and averaging over $\omega(x)$ with a certain class of weights, leads to a one-parameter class of propagators

$$D_F^{\mu\nu}(k) = \frac{-\mathrm{i}}{k^2 + \mathrm{i}\epsilon} \left(\eta^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2} \right) \tag{29}$$

Here $\xi = 0$ is called *Landau gauge*, $\xi = 1$ is called *Feynman gauge*, and $\xi = 3$ is called *Yennie gauge*.

Problem 3. Grassmann integration

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general complex 2×2 matrix, θ_1, θ_2, θ_1^*, θ_2^* independent Grassmann variables, and

$$S_1 = (\theta_1, \theta_2)\, \boldsymbol{A} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{30}$$

$$S_2 = (\theta_1^*, \theta_2^*)\, \boldsymbol{A} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{31}$$

Do the following Grassmann integrals

a)

$$I_1 = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, \mathrm{e}^{S_1}. \tag{32}$$

Using the fact that $\theta_i \, \theta_j = -\theta_j \, \theta_i$ we may simplify the action to $S_1 = (c-b)\, \theta_2 \, \theta_1$. Thus

$$I_1 = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, \mathrm{e}^{S_1} = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, [1 + (c-b)\, \theta_2 \, \theta_1] = c - b, \tag{33}$$

after repeated use of the Grassmann integration rule $\int \mathrm{d}\theta_i (z_0 + z_1 \, \theta_i) = z_1$ (when z_0 and z_1 are ordinary complex numbers).

b)

$$I_2 = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, \theta_1 \theta_2 \, \mathrm{e}^{S_1}. \tag{34}$$

Since $\theta_1 \, \theta_2 \, \mathrm{e}^{(c-b)\, \theta_2 \, \theta_1} = \theta_1 \, \theta_2 = -\theta_2 \, \theta_1$ we find

$$I_2 = -\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, \theta_2 \theta_1 = -1. \tag{35}$$

c)

$$I_3 = \int \mathrm{d}\theta_1^* \, \mathrm{d}\theta_2^* \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; \mathrm{e}^{S_2}. \tag{36}$$

We may simplify the action to $S_2 = a \, \theta_1^* \, \theta_1 + b \, \theta_1^* \, \theta_2 + c \, \theta_2^* \, \theta_1 + d \, \theta_2^* \, \theta_2$, i.e.

$$\mathrm{e}^{S_2} = \mathrm{e}^{a \, \theta_1^* \, \theta_1} \, \mathrm{e}^{b \, \theta_1^* \, \theta_2} \, \mathrm{e}^{c \, \theta_2^* \, \theta_1} \, \mathrm{e}^{d \, \theta_2^* \, \theta_2}$$

Expanding the exponentials give $2^4 = 16$ terms; only those proportional to $\theta_2 \, \theta_1 \, \theta_2^* \, \theta_1^*$ are of interest, since all other terms will integrate to zero. Thus we find

$$\begin{aligned}
\mathrm{e}^{S_2} &= \mathrm{e}^{a \, \theta_1^* \, \theta_1} \, \mathrm{e}^{b \, \theta_1^* \, \theta_2} \, \mathrm{e}^{c \, \theta_2^* \, \theta_1} \, \mathrm{e}^{d \, \theta_2^* \, \theta_2} \\
&= a \, d \, \theta_1^* \, \theta_1 \, \theta_2^* \, \theta_2 + b \, c \, \theta_1^* \, \theta_2 \, \theta_2^* \, \theta_1 + \text{ uinteresting terms} \\
&= -(a \, d - b \, c) \, \theta_2 \, \theta_1 \, \theta_2^* \, \theta_1^* + \text{ uinteresting terms} \, .
\end{aligned}$$

Hence we find

$$\begin{aligned}
I_3 &= \int \mathrm{d}\theta_1^* \, \mathrm{d}\theta_2^* \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; \mathrm{e}^{S_2} \\
&= -(a \, d - b \, c) \int \mathrm{d}\theta_1^* \, \mathrm{d}\theta_2^* \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; \theta_2 \, \theta_1 \, \theta_2^* \, \theta_1^* \\
&= -(a \, d - b \, c) = -\det \boldsymbol{A}.
\end{aligned} \tag{37}$$

d)

$$I_4 = \int \mathrm{d}\theta_1^* \, \mathrm{d}\theta_2^* \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \; \theta_1^* \, \theta_2 \, \mathrm{e}^{S_2}. \tag{38}$$

Here the only interesting term in the expansion of e^{S_2} is the one proportional to $\theta_2^*\,\theta_1$. I.e., we may write

$$\theta_1^*\,\theta_2\,\mathrm{e}^{S_2} = c\,\theta_1^*\,\theta_2\,\theta_2^*\,\theta_1 + \text{ uninteresting terms } = c\,\theta_2\,\theta_1\,\theta_2^*\,\theta_1^* + \text{ uninteresting terms},$$

to find

$$I_4 = \int \mathrm{d}\theta_1^*\,\mathrm{d}\theta_2^*\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\;\theta_1^*\,\theta_2\,\mathrm{e}^{S_2} = c. \tag{39}$$