Eksamination FY8104 Symmetry in physics
Wednesday December 9, 2009
Solutions

1a) The order of a subgroup must be a factor of 12, either 1, 2, 3, 4, 6 or 12.

$\{e\}$ is a subgroup of order 1.

Subgroups of order 2, all cyclic, are $\{e, a\}$, $\{e, b\}$, $\{e, c\}$.

Subgroups of order 3, also cyclic, are $\{e, s, w\}$, $\{e, t, z\}$, $\{e, u, x\}$, $\{e, v, y\}$.

There is one subgroup of order 4, $\{e, a, b, c\}$.

Finally, G itself is a subgroup of order 12.

A cyclic group of order n is isomorphic to $\mathbf{Z}_n$, the addition group of integers modulo n.

1b) Conjugation classes are: $C_1 = \{e\}$, $C_2 = \{a, b, c\}$, $C_3 = \{s, t, u, v\}$, $C_4 = \{w, x, y, z\}$.

For example: $sas^{-1} = saw = uw = b$, $sbs^{-1} = sbw = vw = c$.

And: $asa^{-1} = asa = ta = v$, $bsb^{-1} = bsb = ub = t$, $csc^{-1} = csc = vc = u$.

1c) We see that $H = \{e, a, b, c\}$ is a normal subgroup, since it is a union of conjugation classes, $H = C_1 \cup C_2$. It is the only normal subgroup, apart from $\{e\}$ and G.

Its (simultaneously left and right) cosets are $eH = He = H$, $sH = Hs = C_3$ and $wH = Hw = C_4$.

The group elements of the quotient group G/H are the three cosets of H. The fact alone that G/H is of order 3 (a prime number) proves that it is a cyclic group. H is the unit element of G/H, and $(sH)^2 = s^2H = wH$, $(sH)^3 = s^3H = eH = H$.

This is the multiplication table of G/H:

	H	C_3	C_4
H	H	C_3	C_4
C_3	C_3	C_4	H
C_4	C_4	H	C_3

The multiplication table of G as presented in the problem text, with the three cosets of H grouped together, shows directly the multiplication table of G/H.

1d) There are 4 conjugation classes, and hence 4 irreducible representations.

We know the trivial representation $g \mapsto 1$ for every $g \in G$.

One orthogonality relation is the sum of squares of the dimensions,
$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 12$.

Since $n_1 = 1$, the unique solution is $n_1 = n_2 = n_3 = 1$, $n_3 = 3$.

The quotient group G/H is cyclic and Abelian, and all its irreducible representations are one dimensional. To find them, assume that $sH \mapsto x$, where x is an unknown complex number. Then $wH = (sH)^2 \mapsto x^2$ and $H = (sH)^3 \mapsto x^3$.

Since H is the unit element of G/H, in a one dimensional representation it must be represented by the number 1, hence we must have $x^3 = 1$. There are three solutions: the trivial representation $x = 1$, and the two representations $x = \omega$ and $x = \omega^2$, where

$$\omega = \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}} = \frac{-1 + \mathrm{i}\sqrt{3}}{2} \,, \qquad \omega^2 = \omega^{-1} = \mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}} = \frac{-1 - \mathrm{i}\sqrt{3}}{2} \,.$$

How to solve the equation $x^3 = 1$? Write it as

$$x^3 - 1 = (x-1)(x^2 + x + 1) = 0 \,.$$

The two roots $x \neq 1$ are roots of the equation $x^2 + x + 1 = 0$.

This gives us the three one dimensional characters of G/H, which are immediately three one dimensional characters of G. The fourth character of G is then found from the orthogonality relation which says that columns 2, 3, 4 of the character table are orthogonal to column 1.

Character table (number of elements of each conjugation class in parenthesis):

	$C_1(1)$	$C_2(3)$	$C_3(4)$	$C_4(4)$
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

1e) The rotation angle α is 0 for the unit element e and $\pi = 180^\circ$ for the second order elements a, b, c. Either it is $2\pi/3 = 120^\circ$ for the third order elements s, t, u, v and $4\pi/3 = 240^\circ$ for w, x, y, z. Or it is $4\pi/3 = 240^\circ$ for s, t, u, v and $8\pi/3$, equivalent to $2\pi/3 = 120^\circ$, for w, x, y, z.

The dimension of the $\ell = 2$ representation of SO(3) is

$$\lim_{\alpha \to 0} \chi^{(\ell)}(\alpha) = \lim_{\alpha \to 0} \frac{\sin((\ell + \frac{1}{2})\alpha)}{\sin(\frac{\alpha}{2})} = 2\ell + 1 = 5 \,.$$

Furthermore, we have that

$$\chi^{(2)}(\pi) = \frac{\sin(\frac{5\pi}{2})}{\sin(\frac{\pi}{2})} = 1 \,,$$

$$\chi^{(2)}\left(\frac{2\pi}{3}\right) = \frac{\sin(\frac{5\pi}{3})}{\sin(\frac{\pi}{3})} = \frac{\sin(-\frac{\pi}{3})}{\sin(\frac{\pi}{3})} = -1 \,, \qquad \chi^{(2)}\left(\frac{4\pi}{3}\right) = \frac{\sin(\frac{10\pi}{3})}{\sin(\frac{2\pi}{3})} = \frac{\sin(-\frac{2\pi}{3})}{\sin(\frac{2\pi}{3})} = -1 \,.$$

Thus, the character is

	$C_1(1)$	$C_2(3)$	$C_3(4)$	$C_4(4)$
χ	5	1	-1	-1

The square sum of the character values is $5^2 + 3 \times 1^2 + 8 \times (-1)^2 = 36 = 3 \times 12$.

Thus, the multiplicities m_1, m_2, m_3, m_4 of the four irreducible representations of A_4 are such that $m_1^2 + m_2^2 + m_3^2 + m_4^2 = 3$. We can tell from this that three irreducible representations occur with multiplicity $m = 1$ and the fourth is absent.

To get dimension 5 we must include the representation of dimension 3 and two representations of dimension 1. We have to choose the two non-trivial one dimensional representations in order to get character values that are real (not complex). Thus, $\chi^{(\ell=2)} = \chi_2 + \chi_3 + \chi_4$, as we can verify.

Of course, we may use the orthogonality relations to determine the multiplicity of each irreducible representation. For example,

$$\begin{aligned}(\chi_2, \chi) &= 1 \times 5 + 3 \times 1 \times 1 + 4 \times \omega^* \times (-1) + 4 \times (\omega^2)^* \times (-1) \\ &= 5 + 3 - 4(\omega^2 + \omega) = 5 + 3 + 4 = 12 ,\end{aligned}$$

which shows that the multiplicity of χ_2 is 1.

2a) We have to compute $a(a^\dagger)^n |0\rangle$. We want to commute the operator a to the right, using the commutation relation $[a, a^\dagger] = 1$, or $aa^\dagger = a^\dagger a + 1$, and then use that $a|0\rangle = 0$. One way to do it is as follows,

$$a(a^\dagger)^n |0\rangle = (a(a^\dagger)^n - (a^\dagger)^n a) |0\rangle = [a, (a^\dagger)^n] |0\rangle .$$

Using the Leibniz rule repeatedly we get that

$$[a, (a^\dagger)^n] = [a, a^\dagger](a^\dagger)^{n-1} + a^\dagger [a, a^\dagger](a^\dagger)^{n-2} + \cdots + (a^\dagger)^{n-1}[a, a^\dagger] = n(a^\dagger)^{n-1} .$$

Hence,

$$a|n\rangle = \frac{1}{\sqrt{n!}} a(a^\dagger)^n |0\rangle = \frac{1}{\sqrt{n!}} n(a^\dagger)^{n-1} |0\rangle = \frac{\sqrt{n}}{\sqrt{(n-1)!}} (a^\dagger)^{n-1} |0\rangle = \sqrt{n}\, |n-1\rangle .$$

Since

$$a^\dagger |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^{n+1} |0\rangle = \frac{\sqrt{n+1}}{\sqrt{(n+1)!}} (a^\dagger)^{n+1} |0\rangle = \sqrt{n+1}\, |n+1\rangle ,$$

we have that

$$N|n\rangle = a^\dagger a |n\rangle = \sqrt{n}\, a^\dagger |n-1\rangle = n|n\rangle .$$

We use n times the relation $a(a^\dagger)^k |0\rangle = k(a^\dagger)^{k-1} |0\rangle$ to deduce that

$$\begin{aligned}\langle n|n\rangle &= \frac{1}{n!} \langle 0|a^n (a^\dagger)^n|0\rangle = \frac{n}{n!} \langle 0|a^{n-1}(a^\dagger)^{n-1}|0\rangle = \frac{n(n-1)}{n!} \langle 0|a^{n-2}(a^\dagger)^{n-2}|0\rangle \\ &= \ldots = \frac{n!}{n!} \langle 0|0\rangle = 1 .\end{aligned}$$

2b) Proof that the operator $D = \mathrm{e}^{za^\dagger - z^* a}$ is unitary: $D^\dagger = \mathrm{e}^{(za^\dagger - z^* a)^\dagger} = \mathrm{e}^{z^* a - za^\dagger} = D^{-1}$.

We have that

$$\begin{aligned}DaD^{-1} &= a + [za^\dagger - z^* a, a] + \frac{1}{2} [za^\dagger - z^* a, [za^\dagger - z^* a, a]] + \cdots \\ &= a - z - \frac{1}{2} [za^\dagger - z^* a, z] + \cdots = a - z .\end{aligned}$$

The complex numbers z and z^* commute with everything, and we have the commutation relations $[a, a] = 0$ and $[a^\dagger, a] = -[a, a^\dagger] = -1$.

We have also that

$$(a - z)|z\rangle = DaD^{-1}D|0\rangle = Da|0\rangle = 0 .$$

Some candidates used a different trick for computing DaD^{-1}. In a similar way as above we may show that

$$[a, (za^\dagger - z^* a)^n] = nz\, (za^\dagger - z^* a)^{n-1} \ .$$

Writing out the power series defining $D^{-1} = \mathrm{e}^{-(za^\dagger - z^* a)}$, we then find that

$$[a, D^{-1}] = -z\, D^{-1} \ .$$

Hence,

$$DaD^{-1} = DD^{-1}a + D\,[a, D^{-1}] = a - z \ .$$

2c) Let us use first the second method indicated. The Campbell–Baker–Hausdorff formula simplifies to

$$\mathrm{e}^{za^\dagger}\mathrm{e}^{-z^* a} = \mathrm{e}^{za^\dagger - z^* a + \frac{1}{2}\,[za^\dagger, -z^* a]} = \mathrm{e}^{za^\dagger - z^* a + \frac{1}{2}\,|z|^2} \ ,$$

since all the higher order commutators vanish. Thus,

$$D = \mathrm{e}^{za^\dagger - z^* a} = \mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{e}^{za^\dagger}\mathrm{e}^{-z^* a} \ .$$

Using the power series expansion of the exponentials we get that

$$\mathrm{e}^{-z^* a}\,|0\rangle = \left(I - z^* a + \frac{1}{2}\,(-z^* a)^2 + \cdots\right)|0\rangle = |0\rangle \ ,$$

and

$$\begin{aligned}
\mathrm{e}^{za^\dagger}\,|0\rangle &= \left(I + za^\dagger + \frac{1}{2}\,(za^\dagger)^2 + \cdots + \frac{1}{n!}\,(za^\dagger)^n + \cdots\right)|0\rangle \\
&= |0\rangle + z|1\rangle + \frac{z^2}{2}\,\sqrt{2}\,|2\rangle + \cdots + \frac{z^n}{n!}\,\sqrt{n!}\,|n\rangle + \cdots \\
&= |0\rangle + z|1\rangle + \frac{z^2}{\sqrt{2}}\,|2\rangle + \cdots + \frac{z^n}{\sqrt{n!}}\,|n\rangle + \cdots \ .
\end{aligned}$$

This proves the wanted result,

$$|z\rangle = D\,|0\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\mathrm{e}^{za^\dagger}\mathrm{e}^{-z^* a}\,|0\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle \ .$$

Let us verify that this is an eigenstate of a:

$$a\,|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,a\,|n\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}}\,\sqrt{n}\,|n-1\rangle = \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{k=1}^{\infty}\frac{z^{k+1}}{\sqrt{k!}}\,|k\rangle = z\,|z\rangle \ ,$$

where we define $k = n - 1$.

Since D is a unitary operator and the ground state $|0\rangle$ is normalized, the coherent state $|z\rangle = D\,|0\rangle$ has to be normalized, but let us verify that it actually is:

$$\langle z|z\rangle = \mathrm{e}^{-|z|^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(z^*)^m z^n}{\sqrt{m!n!}}\,\langle m|n\rangle \ .$$

The energy eigenstates are orthonormal, $\langle m|n\rangle = \delta_{mn}$. Orthogonality is a general property of eigenvectors with different eigenvalues of a Hermitean operator such as the Hamiltonian H or the number operator $N = a^\dagger a$. Orthogonality may also be proved directly by the methods used under point 2a) above. It follows that the double sum reduces to a single sum,

$$\langle z|z\rangle = \mathrm{e}^{-|z|^2} \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = 1 \ .$$

2d) For the energy eigenstates we have that

$$U|n\rangle = U(t)|n\rangle = \mathrm{e}^{-\mathrm{i}tH}|n\rangle = \mathrm{e}^{-\mathrm{i}t(n+\frac{1}{2})}|n\rangle \ . \tag{1}$$

Hence,

$$U|z\rangle = \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} U|n\rangle = \mathrm{e}^{-\mathrm{i}\frac{t}{2}} \mathrm{e}^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{(z\mathrm{e}^{-\mathrm{i}t})^n}{\sqrt{n!}} |n\rangle = \mathrm{e}^{-\frac{\mathrm{i}t}{2}} |\mathrm{e}^{-\mathrm{i}t} z\rangle \ .$$

The alternative method for solving the problem is to compute

$$UaU^{-1} = a + [-\mathrm{i}tH, a] + \frac{1}{2}[-\mathrm{i}tH, [-\mathrm{i}tH, a]] + \cdots \ .$$

With $H = a^\dagger a + \frac{1}{2}$ we get that

$$[H, a] = [a^\dagger a, a] = [a^\dagger, a]\, a = -a \ ,$$

and hence,

$$UaU^{-1} = a + \mathrm{i}ta + \frac{1}{2}(\mathrm{i}t)^2 a + \cdots + \frac{1}{n!}(\mathrm{i}t)^n a + \cdots = \mathrm{e}^{\mathrm{i}t} a \ .$$

By Hermitean conjugation, using that $U^{-1} = U^\dagger$, we get that

$$Ua^\dagger U^{-1} = (UaU^{-1})^\dagger = \mathrm{e}^{-\mathrm{i}t} a^\dagger \ .$$

Hence,

$$UDU^{-1} = U(t)\, D(z)\, (U(t))^{-1} = \mathrm{e}^{zUa^\dagger U^{-1} - z^* UaU^{-1}} = \mathrm{e}^{\mathrm{e}^{-\mathrm{i}t} z a^\dagger - \mathrm{e}^{\mathrm{i}t} z^* a} = D(\mathrm{e}^{-\mathrm{i}t} z) \ .$$

Putting these results together we get that

$$U|z\rangle = UD|0\rangle = UDU^{-1}U|0\rangle = \mathrm{e}^{-\frac{\mathrm{i}t}{2}} (UDU^{-1})|0\rangle = \mathrm{e}^{-\frac{\mathrm{i}t}{2}} |\mathrm{e}^{-\mathrm{i}t} z\rangle \ .$$

As a side remark, equation (1) has an interesting consequence. The time development operator $U = U(t)$ with $t = 2\pi$ takes the energy eigenstate $|n\rangle$ into

$$U(2\pi)|n\rangle = \mathrm{e}^{-\mathrm{i}2\pi(n+\frac{1}{2})}|n\rangle = -|n\rangle \ .$$

Since every state of the oscillator may be expanded as a linear combination of energy eigenstates, this proves that $U(2\pi) = -I$. Multiplication of any state vector by an overall phase factor, -1 in this case, does not change the physical state. Hence, all states of the harmonic oscillator are periodic with period 2π (in the units used here).

Another interesting observation is that

$$U(\pi)|n\rangle = \mathrm{e}^{-\mathrm{i}\pi(n+\frac{1}{2})}|n\rangle = -\mathrm{i}(-1)^n |n\rangle \ .$$

In other words, $\mathrm{i}U(\pi)$ is the parity operator, $+1$ for $n = 0, 2, 4, \ldots$ (even parity), and -1 for $n = 1, 3, 5, \ldots$ (odd parity).

2e) Using the commutation relations

$$\begin{aligned}
[H,x] &= \frac{1}{2}\,[p^2,x] = \frac{1}{2}\,([p,x]\,p + p\,[p,x]) = -\mathrm{i}p\ ,\\
[H,p] &= \frac{1}{2}\,[x^2,p] = \frac{1}{2}\,([x,p]\,x + x\,[x,p]) = \mathrm{i}x\ ,
\end{aligned}$$

we get that

$$\begin{aligned}
UxU^{-1} &= x + [-\mathrm{i}tH,x] + \frac{1}{2}\,[-\mathrm{i}tH,[-\mathrm{i}tH,x]] + \frac{1}{3!}\,[-\mathrm{i}tH,[-\mathrm{i}tH,[-\mathrm{i}tH,x]]] + \cdots\\
&= x - tp - \frac{t^2}{2}\,x + \frac{t^3}{3!}\,p + \cdots = (\cos t)\,x - (\sin t)\,p\ ,
\end{aligned}$$

and

$$\begin{aligned}
UpU^{-1} &= p + [-\mathrm{i}tH,p] + \frac{1}{2}\,[-\mathrm{i}tH,[-\mathrm{i}tH,p]] + \frac{1}{3!}\,[-\mathrm{i}tH,[-\mathrm{i}tH,[-\mathrm{i}tH,p]]] + \cdots\\
&= p + tx - \frac{t^2}{2}\,p - \frac{t^3}{3!}\,x + \cdots = (\cos t)\,p + (\sin t)\,x\ .
\end{aligned}$$

In consequence,

$$UbU^{-1} = UxU^{-1} + \mathrm{i}\lambda UpU^{-1} = (\cos t + \mathrm{i}\lambda\sin t)\,x + (-\sin t + \mathrm{i}\lambda\cos t)\,p\ .$$

We now assume that the state at time $t = 0$ is the squeezed vacuum state, that

$$|\psi(0)\rangle = |\lambda,0\rangle\ .$$

The state at time t is $|\psi(t)\rangle = U\,|\psi(0)\rangle$ with $U = U(t) = \mathrm{e}^{-\mathrm{i}tH}$.

From the equation $b\,|\psi(0)\rangle = 0$ follows that

$$UbU^{-1}\,|\psi(t)\rangle = UbU^{-1}U\,|\psi(0)\rangle = Ub\,|\psi(0)\rangle = 0\ .$$

If we choose the time t to be one half of the period of the oscillator, $t = \pi$, then $UbU^{-1} = -b$, and we see that the state $|\psi(\pi)\rangle$ is again the same squeezed vacuum state as at time $t = 0$.

More interesting is what happens after one quarter period, at $t = \pi/2$. Then $\cos t = 0$, $\sin t = 1$, and

$$UbU^{-1} = \mathrm{i}\lambda\,x - p = \mathrm{i}\lambda\left(x + \frac{\mathrm{i}}{\lambda}\,p\right)\ .$$

This shows that $|\psi(\pi/2)\rangle$ is a different squeezed vacuum state, with squeezing parameter $1/\lambda$ instead of λ.