NTNU

Institutt for fysikk

Contact during the exam:
Roger Sollie
Telephone: 95 10 98 05

Exam in FY8304 MATHEMATICAL APPROXIMATION METHODS IN PHYSICS

Wednesday december 17, 2008
09:00–14:00

Allowed help: Alternativ **C**
Standard calculator
K. Rottman: *Matematisk formelsamling* (all languages).
Schaum's Outline Series: *Mathematical Handbook of Formulas and Tables.*

This problem set consists of 3 pages.

Problem 1.
Consider the differential equation

$$x\,y'''(x) + 2\,y(x) \;=\; 0.$$

a) Find and classify the singular points of this equation.

b) Find the controling factor $y_c(x)$ for a decaying solution as $x \to \infty$.

c) The leading asymptotic behavior (as $x \to \infty$) can be written on the form

$$y(x) \sim A\,x^{\alpha}\,y_c(x)\ .$$

Use the method of dominant balance to determine the exponent α.

d) Show that the function

$$y(x) \;=\; \int_0^{\infty} \mathrm{e}^{-t-\frac{x}{\sqrt{t}}}\,\mathrm{d}t$$

satisfies the differential equation.

e) Use Laplace's method on the integral to determine the constant A.

Problem 2.

The differential equation

$$\dot{y} = y^2 - y\,,$$

describes the time dependence of a population $y(t)$ with a quadratic birth rate (y^2) and a linear death rate ($-y$).

a) What are the critical points of the equation?

b) Classify the critical points by finding the local behavior near them, and draw the phase line (one-dimensional phase space).

c) Based on the above analysis, make a sketch showing (qualitatively) the behavior of $y(t)$ for various initial values $y(0)$.

d) Find the analytic solution $y(t)$ of the equation for a given initial value $y(0)$

e) What happens after the time t defined by

$$e^t = \frac{y(0)}{y(0)-1}\ ?$$

How does this modify the sketch made in **c**)?

Problem 3.

Consider the boundary value problem

$$\varepsilon y''(x) + (1+x)y'(x) + ay(x) = 0, \quad y(0) = y(1) = 1,$$

in the limit $\varepsilon \to 0^+$.

a) Find the outer solution to the boundary value problem.

b) Determine the position of the boundary layer, and how its thickness scale with ε.

c) Find the inner solution to the boundary value problem.

d) Write down the uniform solution to the boundary value problem.

e) Assume that the boundary value problem instead is

$$\varepsilon y''(x) + xy'(x) + ay(x) = 0, \quad y(0) = y(1) = 1.$$

Find the inner equation, i.e. the equation which describes the boundary layer, for this problem.

Problem 4.

Consider the oscillator described by

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \omega^2(\varepsilon t)\, y(t) = 0\,,$$

where the frequency $\omega(\varepsilon t)$ is a slowly varying function of time t (i.e., small positive values of the perturbation parameter ε).

a) Define a new time scale $\tau = \varepsilon t$, and show why a multiple-scale expansion

$$y(t) = Y_0(t,\tau) + \varepsilon Y_1(t,\tau) + \ldots\,,$$

in this case fails.

b) Introduce another time variable $T = f(t)$, and determine $f(t)$ by requiring that the frequency of the unperturbed oscillator is constant.

c) Show how one can use multiple-scale analysis to recover the WKB solution of the oscillator equation.