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Løsningsforslag til eksamen i FY8307/3404/TFY18 RELATIVISTISK KVANTEMEKANIKK

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Skriv *studentnummeret* ditt på hvert ark av oppgavesettet.

Dette løsningsforslaget er på 5 sider.

Oppgave 1.
En variant av fri elektrodynamikk er definert ved Lagrangetettheten

$$\mathcal{L} = \frac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) + \theta \boldsymbol{E}\cdot\boldsymbol{B} \tag{1}$$

der $\boldsymbol{E} = -\dot{\boldsymbol{A}}$ og $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$.
I denne modellen er den kanonisk konjugerte impulstettheten til $\boldsymbol{A}$ gitt som

A.	$\Pi_{\boldsymbol{A}} = \dot{\boldsymbol{A}}$	
B.	$\Pi_{\boldsymbol{A}} = \boldsymbol{E}$	
C.	$\Pi_{\boldsymbol{A}} = \boldsymbol{E} + \theta\,\boldsymbol{B}$	
D.	$\Pi_{\boldsymbol{A}} = -\boldsymbol{E} - \theta\,\boldsymbol{B}$	X
E.	$\Pi_{\boldsymbol{A}} = \dot{\boldsymbol{A}} + \theta\,\boldsymbol{\nabla}\cdot\boldsymbol{A}$	

Replacing $\boldsymbol{E}$ by $-\dot{\boldsymbol{A}}$ the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left(\dot{\boldsymbol{A}}^2 - \boldsymbol{B}^2\right) - \theta\dot{\boldsymbol{A}}\cdot\boldsymbol{B} \tag{2}$$

from which it is easy to see that $\Pi_{\boldsymbol{A}} = \frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{A}}} = \dot{\boldsymbol{A}} - \theta\boldsymbol{B} = -\boldsymbol{E} - \theta\boldsymbol{B}$.

Oppgave 2.
Bevegelsesligningen som kan utledes fra Lagrangetettheten (1) er

A.	$\Box\boldsymbol{A} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A}\right) = 0$	X
B.	$\Box\boldsymbol{A} + \theta\,\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$	
C.	$\Box\boldsymbol{A} + \theta\left(\frac{\partial}{\partial t}\boldsymbol{B} - \boldsymbol{\nabla}\times\boldsymbol{E}\right) = 0$	
D.	$\Box\boldsymbol{A} = 0$	
E.	$\Box\boldsymbol{A} + \theta\left(\boldsymbol{E} - \boldsymbol{B}\right) = 0$	

Oppgitt: $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{X}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{X}) - \Delta \boldsymbol{X}$.

Here the challenge is to handle the $\boldsymbol{B}^2$ term. We may write $B^i = \varepsilon^{imn}\partial_m A^n$, so that

$$\boldsymbol{B}^2 = \varepsilon^{imn}\partial_m A^n \varepsilon^{ipq}\partial_p A^q \qquad \text{and} \qquad \boldsymbol{E} \cdot \boldsymbol{B} = -\dot{A}^i \, \varepsilon^{imn}\partial_m A^n.$$

Then we find that

$$\begin{aligned} \partial_j \frac{\partial \boldsymbol{B}^2}{\partial(\partial_j A^k)} &= \partial_j \left(\varepsilon^{ijk}\varepsilon^{ipq}\partial_p A^q + \varepsilon^{imn}\partial_m A^n \varepsilon^{ijk} \right) = -2\varepsilon^{kji}\partial_j B^i \\ &= -2\left(\boldsymbol{\nabla} \times \boldsymbol{B}\right)^k = -2\left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\right]^k, \end{aligned}$$

and

$$\partial_j \frac{\partial \boldsymbol{E} \cdot \boldsymbol{B}}{\partial(\partial_j A^k)} = -\partial_j \dot{A}^i \, \varepsilon^{ijk} = \varepsilon^{kji}\partial_j \dot{A}^i = \dot{B}^k,$$

so that

$$\partial_j \frac{\mathcal{L}}{\partial(\partial_j A^k)} = -\left(\boldsymbol{\nabla} \times \boldsymbol{B}\right)^k + \theta \dot{B}^k = \left[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\right]^k + \theta \dot{B}^k.$$

Likewise

$$\partial_t \frac{\mathcal{L}}{\partial \dot{A}^k} = \partial_t \Pi_{A^k} = \ddot{A}^k - \theta \dot{B}.$$

Thus, the Euler Lagrange equations become

$$\partial_t \frac{\mathcal{L}}{\partial \dot{A}^k} + \partial_j \frac{\mathcal{L}}{\partial(\partial_j A^k)} = \frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \Box \boldsymbol{A} - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) = 0 \tag{3}$$

Oppgave 3.

Målt i naturlige enheter er positronets ladning omtrent lik

A.	1.6×10^{-19}	
B.	0.0073	
C.	0.303	X
D.	0.085	
E.	-1.6×10^{-19}	

The fine structure constant is

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137.04}. \tag{4}$$

Thus, in natural units ($\hbar = c = \varepsilon_0 = \mu_0 = 1$) we have $e = \sqrt{4\pi\alpha} \approx \sqrt{4\pi/137.04} \approx 0.303$.

Oppgave 4.

Basert på ren dimensjonsanalyse ville man (i naturlige) enheter anslå at vakuumenergien er av størrelsesorden $\varepsilon \approx 1\ \text{TeV}^4 = \left(10^{12}\ \text{eV}\right)^4$. Omregnet til SI-enheter svarer dette til en massetetthet på

A.	$\varepsilon \approx 2.3 \cdot 10^{10}\ \text{kg/m}^3$	
B.	$\varepsilon \approx 2.3 \cdot 10^{16}\ \text{kg/m}^3$	
C.	$\varepsilon \approx 2.3 \cdot 10^{32}\ \text{kg/m}^3$	X
D.	$\varepsilon \approx 2.3 \cdot 10^{48}\ \text{kg/m}^3$	
E.	$\varepsilon \approx 2.3 \cdot 10^{64}\ \text{kg/m}^3$	

Oppgitt: Lyshastigheten $c = 299\,792\,458$ m/s, Planck's konstant $\hbar = 1.055 \cdot 10^{-34}$ kg m^2/s, positronladningen $e = 1.602 \cdot 10^{-19}$ C.

We first convert to proper SI units, using the fact that 1 eV $= 1.603 \cdot 10^{-19}$ J. Now, to convert from an expression with dimension kg^4 m^8/s^8 to something with dimension kg/m^3 we must divide by $\hbar^A c^B$ for appropriate A and B. To get the right dimension of kg we need $A = 3$, Next, to eliminate s we need $B = 5$. This leaves the correct powers of m, $8 - 3 \times 2 - 5 = -3$. Thus, the SI value of our expression is

$$\varepsilon \approx \left(10^{12} \times 1.604 \cdot 10^{19}\right)^4 / ((1.055 \cdot 10^{-34})^3 (299\,792\,458)^5)\ \frac{\text{kg}}{\text{m}^3} \approx 2.3 \cdot 10^{-32}\ \text{kg/m}^3. \quad (5)$$

Oppgave 5.
Vi definerer matrisen $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Da vil

A.	γ^5 kommutere med alle matrisene γ^μ	
B.	γ^5 kommutere med γ^0 og antikommutere med alle γ^i	
C.	γ^5 antikommutere med alle matrisene γ^μ	X
D.	γ^5 antikommutere med γ^0 og kommutere med alle γ^i	
E.	γ^5 kommutere med γ^0, γ^2 og antikommutere med γ^1, γ^3	

Consider the matrix γ^μ for any $\mu = 0, \ldots, 3$. To change $\gamma^\mu\gamma^5$ into $\gamma^5\gamma^\mu$ we have to anticommute through three γ-matrices (those with index different from μ) and commute through one (the one with index μ). Altogether we get an odd number of minus-signs, i.e. anticommuting property. Not that this argument holds in all even space-time dimensions.

Oppgave 6.
Feynmans propagator for Dirac-partikler er definert ved ligningen

$$\mathrm{i}S_F(x-y) = \langle\Omega|\mathcal{T}\left\{\psi(x)\bar{\psi}(y)\right\}|\Omega\rangle.$$

der $\mathcal{T}$ er tidsordningsoperatoren. For Dirac-partikler med masse m kan denne propagatoren representeres ved Fourier-integralet (i naturlige enheter)

A.	$\mathrm{i}S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\ \mathrm{e}^{-\mathrm{i}p(x-y)}$	
B.	$\mathrm{i}S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{-\mathrm{i}}{p^2-m^2+\mathrm{i}\epsilon}\ \mathrm{e}^{-\mathrm{i}p(x-y)}$	
C.	$\mathrm{i}S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}(\not{p}+m)}{p^2-m^2+\mathrm{i}\epsilon}\ \mathrm{e}^{-\mathrm{i}p(x-y)}$	X
D.	$\mathrm{i}S_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{\mathrm{i}(\not{p}-m)}{p^2-m^2+\mathrm{i}\epsilon}\ \mathrm{e}^{-\mathrm{i}p(x-y)}$	
E.	Ingen av alternativene over.	

The Fourier transform of the Klein Gordon propagator is

$$\frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

The Fourier transform of the Dirac propagators is

$$\frac{\mathrm{i}\,(\not{p} + m)}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

These are very important expressions in quantum field theory, and should be remembered by heart!

Oppgave 7.
La $\mathcal{T}$ være tidsordningsoperatoren og $\psi(x)$, $\bar{\psi}(x)$ et kvantisert Diracfelt. Da gjelder (i naturlige enheter, dvs. når $\hbar = c = 1$)

A.	$\mathcal{T}\left\{\psi(x)\bar{\psi}(y)\right\} = \mathcal{T}\left\{\bar{\psi}(y)\psi(x)\right\}$	
B.	$\mathcal{T}\left\{\psi(x)\bar{\psi}(y)\right\} = \mathcal{T}\left\{\bar{\psi}(y)\psi(x)\right\} + \mathrm{i}S_F(x-y)$	
C.	$\mathcal{T}\left\{\psi(x)\bar{\psi}(y)\right\} = \mathcal{T}\left\{\bar{\psi}(y)\psi(x)\right\} - \mathrm{i}S_F(x-y)$	
D.	$\mathcal{T}\left\{\psi(x)\bar{\psi}(y)\right\} = -\mathcal{T}\left\{\bar{\psi}(y)\psi(x)\right\} - \mathrm{i}S_F(x-y)$	
E.	Ingen av alternativene over.	X

Her er $S_F(x-y)$ er Feynmans propagator for Dirac-partikler.

The definition of $\mathcal{T}$ is such that fermions *anticommute* under the time ordering symbol.

Oppgave 8.
Gitt feltutviklingen for et fritt elektromagnetisk felt i Coulomb gauge,

$$\boldsymbol{A}(x) = \sum_{\boldsymbol{k},r} \frac{1}{\sqrt{2|\boldsymbol{k}|V}} \left(a_{\boldsymbol{k},r}\, \hat{e}_{\boldsymbol{k},r}\, \mathrm{e}^{-ikx} + \text{hermittisk konjugert} \right). \tag{6}$$

Da er matrise-elementet $\langle\Omega|\, a_{\boldsymbol{q},s}\, \boldsymbol{A}(x)\, |\Omega\rangle$ lik

A.	0	
B.	$\frac{1}{\sqrt{2|\boldsymbol{k}|V}}\, \hat{e}_{\boldsymbol{q},s}\, \mathrm{e}^{-iqx}$	
C.	$a_{\boldsymbol{k},s}$	
D.	$\frac{1}{\sqrt{2|\boldsymbol{k}|V}}\, \hat{e}^*_{\boldsymbol{q},s}\, \mathrm{e}^{iqx}$	X
E.	Ingen av alternativene over.	

To get a nonzero result we must use the creation operator $a^\dagger_{\boldsymbol{q},s}$ from the expansion of $\boldsymbol{A}(x)$; this goes together with the factor $\frac{1}{\sqrt{2|\boldsymbol{k}|V}}\, \hat{e}^*_{\boldsymbol{q},s}\, \mathrm{e}^{iqx}$.

Oppgave 9.
Lagrangefunksjonen for en modell i kvanteelektrodynamikk med et komplekst Klein-Gordon felt og et Dirac-felt lyder

$$\mathcal{L} = \frac{1}{2}\left(D^{(\varphi)}_\mu \varphi\right)^* \left(D^{(\varphi)\,\mu}\varphi\right) - V(\varphi^*\varphi) + \bar{\Psi}\left(\mathrm{i}\gamma^\mu D^{(\psi)}_\mu - m\right)\Psi, \tag{7}$$

der $D^{(\varphi)}_\mu = \partial_\mu + 2\mathrm{i}eA_\mu$ og $D^{(\psi)}_\mu = \partial_\mu - \mathrm{i}eA_\mu$. For bestemte verdier av koeffisientene a og b er denne modellen invariant under lokale gaugetransformasjoner,

$$A_\mu(x) \;\to\; A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x), \tag{8}$$

$$\varphi(x) \;\to\; \varphi'(x) = \mathrm{e}^{\mathrm{i}a\Lambda(x)}\,\varphi(x), \tag{9}$$

$$\Psi(x) \;\to\; \Psi'(x) = \mathrm{e}^{\mathrm{i}b\Lambda(x)}\,\Psi(x). \tag{10}$$

Skriv ned disse verdiene:

$a = \boxed{\text{-2e}}$

$b = \boxed{\text{e}}$

We want

$$D_\mu^{(\varphi)\prime}\,\varphi' = \mathrm{e}^{\mathrm{i}a\Lambda}\,D_\mu^{(\varphi)}\,\varphi,$$

or written out

$$[\partial_\mu + 2\mathrm{ie}\,(A_\mu + \partial_\mu\Lambda)]\,\mathrm{e}^{\mathrm{i}a\Lambda}\varphi = \mathrm{e}^{\mathrm{i}a\Lambda}\,[\partial_\mu + \mathrm{i}a\partial_\mu\Lambda + 2\mathrm{ie}\,(A_\mu + \partial_\mu\Lambda)]\,\varphi = \mathrm{e}^{\mathrm{i}a\Lambda}\,\left(\partial_\mu - \mathrm{ie}A_\mu\right)\varphi,$$

i.e. that $ia\partial_\mu\Lambda + 2\mathrm{i}e\partial_\mu\Lambda = 0$ or $a = -2e$.

We want

$$D_\mu^{(\psi)\prime}\,\Psi' = \mathrm{e}^{\mathrm{i}b\Lambda}\,D_\mu^{(\psi)}\,\Psi,$$

or written out

$$[\partial_\mu - \mathrm{ie}\,(A_\mu + \partial_\mu\Lambda)]\,\mathrm{e}^{\mathrm{i}b\Lambda}\varphi = \mathrm{e}^{\mathrm{i}b\Lambda}\,[\partial_\mu + \mathrm{i}b\partial_\mu\Lambda - \mathrm{ie}\,(A_\mu + \partial_\mu\Lambda)]\,\varphi = \mathrm{e}^{\mathrm{i}b\Lambda}\,\left(\partial_\mu - \mathrm{ie}A_\mu\right)\varphi,$$

i.e. that $ib\partial_\mu\Lambda - \mathrm{i}e\partial_\mu\Lambda = 0$ or $b = e$.

Oppgave 10.

Etter kvantisering med periodiske grensebetingelser i et volum $V = L^3$ ønsker vi å finne integraluttrykk for fysiske størrelser, gyldig i grensen $V \to \infty$. For tilstrekkelig glatte funksjoner $f(\boldsymbol{k})$ skjer dette ved å la

$$\sum_{\boldsymbol{k}} f(\boldsymbol{k}) \to C \int \mathrm{d}^3\boldsymbol{k}\, f(\boldsymbol{k}). \tag{11}$$

Skriv ned størrelsen C:

$$C = \boxed{\frac{V}{(2\pi)^3}}$$

The constants C is the inverse of the volume of the unit cell $\Delta v_k = \left(\frac{2\pi}{L}\right)^d$ of the resiprocal lattice, where d is the space dimension.