Oppgave 1 (cf p. 19 - 22 in textbook)

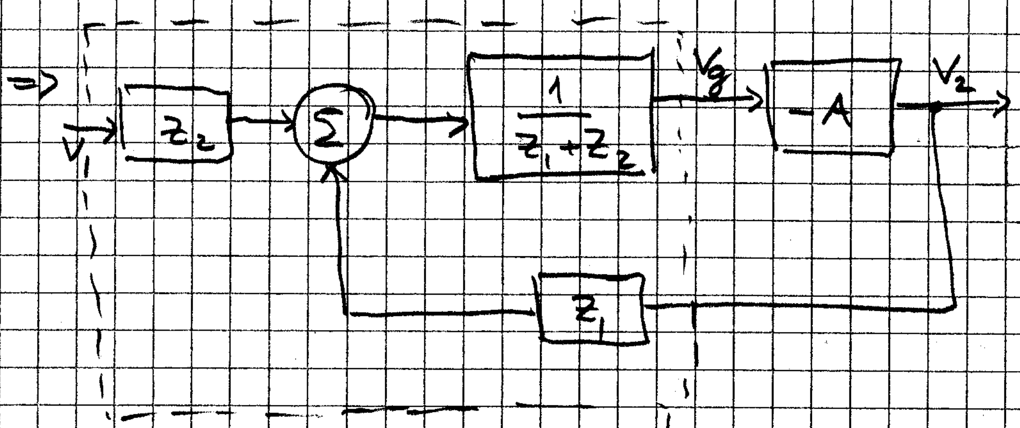
Current i is negligible $\Rightarrow$ $\dfrac{V_1 - V_g}{Z_1} = \dfrac{V_g - V_2}{Z_2}$

or $V_g (Z_1 + Z_2) = V_1 Z_2 + V_2 Z_1$ - - - (1)

with $V_2 = -A(s) \cdot V_g$ - - - (2)

a)

$\Rightarrow$



$\rightarrow$ This block represents eq (1)

b) $G(s) = \dfrac{V_2(s)}{V_1(s)}$

eliminate V_g between (1) and (2) $\Rightarrow$

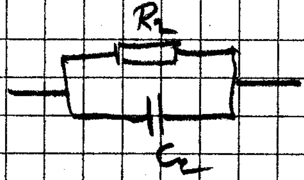
$$-\frac{V_2}{A(s)}\left[Z_1 + Z_2\right] = V_1 Z_2 + V_2 Z_1$$

$$\Rightarrow \quad V_2\left[Z_1 \cdot A(s) + Z_1 + Z_2\right] = -V_1 Z_2 A(s)$$

$$G(s) = \frac{V_2(s)}{V_1(s)} = -\frac{Z_2 A(s)}{Z_1 A(s) + Z_1 + Z_2} = -\frac{Z_2(s)}{Z_1(s)} \cdot \frac{1}{1 + A^{-1} + \frac{Z_2}{Z_1} A^{-1}}$$

When $A(s) \to \infty$ $\quad G(s) = \frac{V_2}{V_1} = -\frac{Z_2}{Z_1} \cdot \frac{1}{1+\varepsilon+\varepsilon} = -\frac{Z_2(s)}{Z_1(s)}$

c)



$$\frac{1}{Z_2} = \frac{1}{R_2} + \frac{1}{1/j\omega C_2} = \frac{\frac{1}{j\omega C_2} + R_2}{R_2 \cdot \frac{1}{j\omega C_2}} = \frac{1 + j\omega C_2 R_2}{R_2}$$

$$Z_2 = \frac{R_2}{1 + j\omega R_2 C_2}$$

$\omega \to 0 \quad Z_2 \to R_2$

$\omega \to \infty \quad Z_2 \to 0$

d)

$$S_A = \left(\frac{dG}{G}\right) \Big/ \left(\frac{dA}{A}\right)$$

$$|G| = \frac{Z_2}{Z_1} \frac{1}{1 + A^{-1} + 100 \cdot A^{-1}} = 100 \cdot \frac{1}{1 + \frac{101}{A}}$$

$$\log G = \log 100 - \log\left(1 + \frac{101}{A}\right) \sim \log 100 - \frac{101}{A}$$

differentiate!

$$\frac{dG}{G} = 0 - 101\left(-\frac{1}{A^2}\right) dA = \frac{101}{A} \cdot \frac{dA}{A}$$

$$S_A = \frac{dG}{G} \Big/ \frac{dA}{A} = \frac{101}{A}$$

Problem 2

$$\begin{cases} \dot{x} = -\alpha x y & (1) \\ \dot{y} = \alpha x y - \beta y & (2) \end{cases}$$

1 => number of healthy people diminishes with rat $\alpha x y$
i.e. probability for infection pr individual is prop.
number of sick persons

$$\dot{x}/x = -\alpha y \qquad \dot{x} = -\alpha x \cdot y$$

2 => number of sick persons increases equal to the number
that becomes sick ($\alpha x y$). Relative death rate is β

$$\frac{\dot{y}}{y} = -\beta$$

$$\dot{y} = \alpha x y - \beta y$$

All persons infected will die, mean life time
will be $1/\beta = \tau$

b) $x = x_0 + x_1$ $\qquad x_1 \ll x_0 \qquad y \ll x \approx x_0$

$$\begin{cases} \dot{x} = \dot{x}_1 = -\alpha(x_0 + x_1) y = -\alpha(x_0) \cdot y \Rightarrow \dot{x}_1 = -\alpha x_0 y \\ \dot{y} = \alpha(x_0 + x_1) y - \beta y = (\alpha x_0 - \beta) y \quad \text{first order diff eq!} \end{cases}$$

$\Rightarrow y = y_0 e^{B(t - t_0)}$ $\qquad$ since $y = y_0$ at $t = t_0$

$$\frac{dy}{dt} = (\alpha x_0 - \beta) \cdot \underbrace{y_0 e^{B(t-t_0)}}_{y} = y_0 B e^{B(t-t_0)}$$

$$\therefore \quad B = (\alpha x_0 - \beta)$$

$$\therefore \quad y = y_0 \, e^{(\alpha x_0 - \beta)(t - t_0)}$$

$$x = x_0 + \int_{t_0}^{t} \dot{x}_1 \, dt = x_0 - \alpha x_0 \int_{t_0}^{t} y \, dt$$

$$= x_0 - \alpha x_0 \left\{ \left[\frac{y_0 e^{(\alpha x_0 - \beta)(t - t_0)}}{(\alpha x_0 - \beta)} \right]_{t = t_0}^{t} \right\}$$

$$= x_0 - \alpha x_0 \left\{ \frac{y_0 e^{(\alpha x_0 - \beta)(t - t_0)}}{(\alpha x_0 - \beta)} - \frac{y_0 \cdot 1}{(\alpha x_0 - \beta)} \right\}$$

$$= x_0 - \frac{\alpha x_0}{(\alpha x_0 - \beta)} \left\{ y_0 e^{(\alpha x_0 - \beta)(t - t_0)} - y_0 \right\}$$

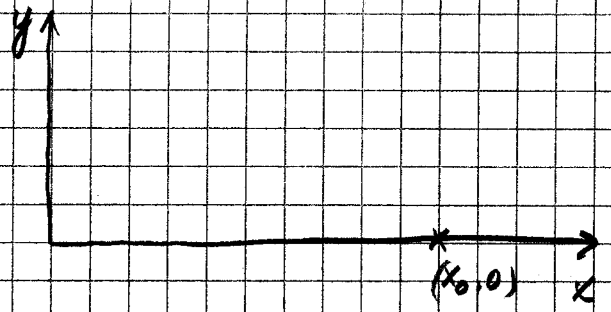
Number of sick persons increases exponentially
—"— —"— healthy persons diminishes —"— !

If $\alpha x_0 = 1 \ (\text{year})^{-1}$ and $\beta = 0.5 \ (\text{year})^{-1}$
[by the way $\tau = \frac{1}{\beta} = 2$ years!]

$$\begin{cases} y = y_0 \, e^{0.5(t - t_0)} \\ x = x_0 - 2 \cdot [y - y_0] = x_0 - 2 y_0 e^{-0.5(t - t_0)} - 2 y_0 \end{cases}$$

(3)

c)



Use isoclines $S = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = -1 + \frac{\beta}{\alpha} \cdot \frac{1}{x} = -1 + \frac{c}{x}$

(since $c \equiv \beta/\alpha$)

$S = 0$ for $x = c = \frac{\beta}{\alpha}$

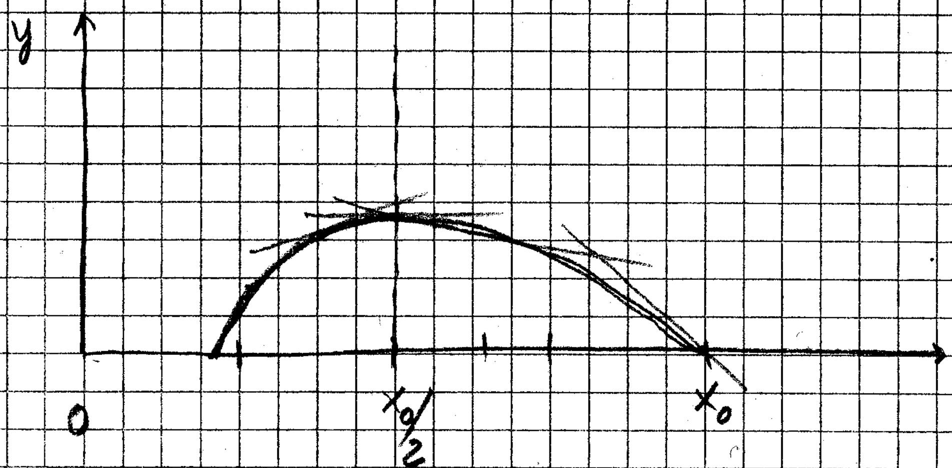
One can write $\frac{dy}{dx} = -1 + \frac{\beta}{\alpha x_0} \cdot \frac{x_0}{x} = -1 + \frac{0.5}{1} \frac{x_0}{x} = S$

using the numbers given in point b)

For $x = x_0$ S will be constant $\left(= -1 + 1/2 = -\frac{1}{2} \right)$

For $x = c$ or $x = \frac{x_0}{2}$ S will be zero

$\alpha x_0 = 1$	$\beta = 0.5$						
x/x_0	1	$2/3$	$1/2$	$4/10$	$3/8$	$1/4$	$3/10$
S	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$1/4$	$1/3$	1	$2/3$



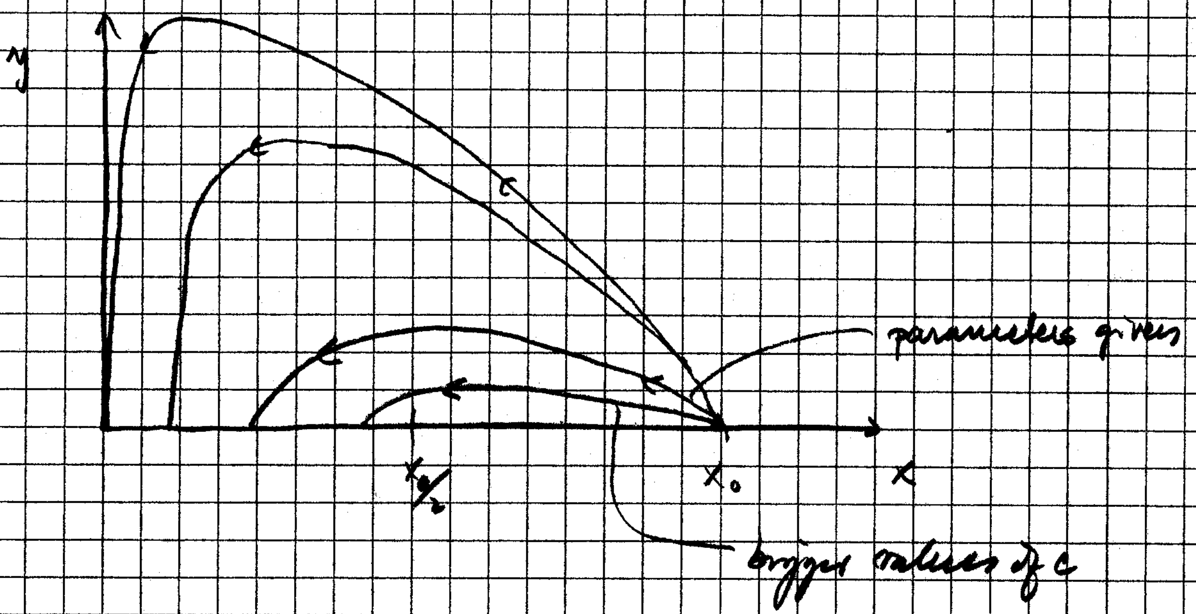
The population does not die out but reaches a low value!

Fixed x_0 varying c (not values used in point b))

If $c > x_0$ $S > 0$ always non physical values.

$$\frac{c}{x_0} = \frac{\beta}{\alpha x_0} \Rightarrow \beta > \alpha x_0 \Rightarrow \dot{y} = \alpha x_0 y - \beta y = (\overbrace{\alpha x_0 - \beta}^{<0}) y$$

$\Rightarrow$ the dying infections does not infect but die immediately (y does not increase)



$S = 0$ for $x = c$

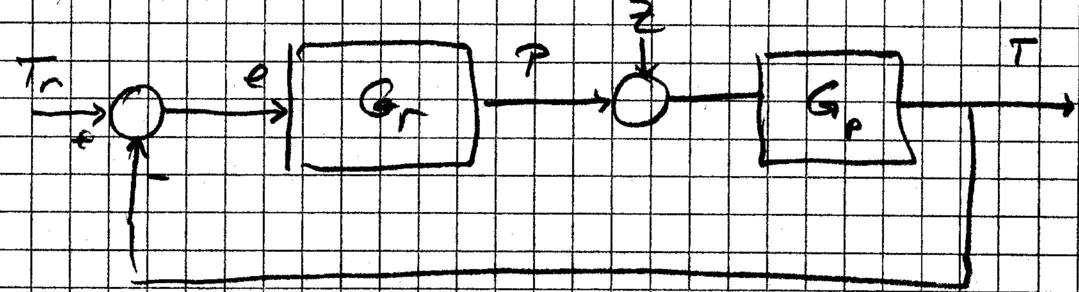
$S \to \infty$ when $x \to 0$

Small value of c: the infectious patient die slowly (β small) they will spread the disease to many persons, the population almost dies out

Oppgave 3

$$C\dot{T} + a(T - T_0) + z(t) = P \qquad (1)$$

Block diagram



T_r reference temp $\qquad e = T_r - T$

Choose $x = T - T_r$ $(= -e)$ as variable

Control strategy $\qquad P = k_1 \int e\,dt + k_2 e \Rightarrow \dot{P} = k_1 e + k_2 \dot{e}$

$$\dot{P} = k_1 e + k_2 \ddot{e} = -(k_1 x + k_2 \dot{x}) \qquad \text{since } \dot{x} = -\dot{e} = \dot{T}$$

Laplace transform eq (1)

$$C(\dot{x} + T_r) + a(x + T_r - T_0) + z(t) = P$$

$$C\dot{x} + ax + a(T_r - T_0) + z(t) = P$$

$$sC\,X(s) + a\,X(s) + a(T_r - T_0) + z_0 - z_0 \frac{s}{s^2+\omega^2} =$$

$$= P(s) = -\left(k_1 \frac{1}{s} X(s) + k_2 X(s)\right)$$

$$\left[sC + a + \frac{1}{s}(k_1 + s k_2)\right] X(s) = \underbrace{-a(T_r - T_0) - z_0}_{\text{not important } = A_0} + z_0 \frac{s}{s^2+\omega^2} \qquad (2)$$

$$F(s) = \frac{z_0 s}{s^2 + \omega^2}$$

from (2):

$$X(s) = \frac{A_0}{[\quad]} + \frac{F(s)}{[\quad]}$$

With $T(s) = \dfrac{s}{[s^2 c + (a+k_2)s + k_1]} = \dfrac{s}{Q(s)}$

we get

$$X(s) = \frac{A_0 \cdot s}{Q(s)} + \frac{F(s)\cdot s}{s[\quad]} = \frac{A_0}{Q(s)} + F(s)\cdot T(s)$$

General solution

$$X(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + z_0 |T(j\omega)| \cos(\omega t + \phi)$$

Disregarding the transients we look at the term

$z_0 |T(j\omega)| \cos(\omega t + \phi)$ with characteristic eq $Q(s)=0$

$Q(s)=0$: $\quad s^2 + \left(\frac{a+k_2}{c}\right) s + \frac{k_1}{c} = 0$

$$\lambda_{1,2} = \frac{1}{2c}\left[-(a+k_2) \pm \sqrt{(a+k_2)^2 - 4k_1 c}\right]$$

System stable when roots in LHP $\quad k_1 > 0$

$a + k_2 > 0$

$$|T(j\omega)| = \frac{\omega}{\sqrt{(k_1 - c\omega^2)^2 + (a+k_2)^2\omega^2}} = \frac{1}{\sqrt{\left(\frac{k_1 - c\omega^2}{\omega}\right)^2 + (a+k_2)^2}}$$

max for min denominator

min when $\frac{k_1 - c\omega^2}{\omega} = 0$

$$\Rightarrow \quad \omega_{max}^2 = \frac{k_1}{c}$$

$$\Rightarrow \quad |T(j\omega)|_{max} = \frac{1}{\sqrt{(a+k_2)^2}} = \frac{1}{a+k_2}$$

for $\omega = \omega_{max} = \sqrt{\frac{k_1}{c}}$

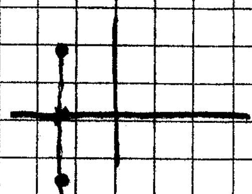
$z_0 |T(j\omega)|_{max} \cdot 1$ is the maximum value

$$(\Delta T)_{max} = 0.1\ K = z_0 |T(j\omega)|_{max} = z_0 / (a+k_2)$$

$$\therefore (k_2 + a) \cdot 0.1 = z_0 \quad \Rightarrow \quad k_2 = \frac{z_0}{0.1} - a$$

$$= \frac{10}{0.1} - 1 = 99\ W/K$$

Time constant should be minimum



$$\Rightarrow \quad (a+k_2)^2 - 4k_1 c \le 0$$

$$k_1 \ge \frac{(a+k_2)^2}{4c} = \frac{(100)^2}{4 \cdot 1000} = \frac{10}{4} = 2.5\ W/K\cdot s$$

$$\Rightarrow \quad \tau = -\frac{1}{\lambda} = \frac{2c}{a+k_2} = \frac{2000}{100} = 20s$$ reasonable!